The Projective Fundamental Group of Hom shifts

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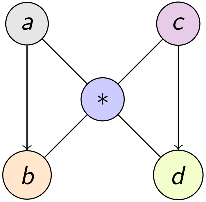
Bi-infinite walks

Consider a directed graph G.
A G-colouring of $\mathbb{Z}$ is a (bi-infinite) walk on G.



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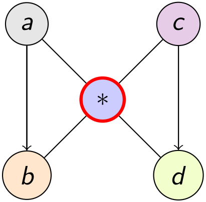

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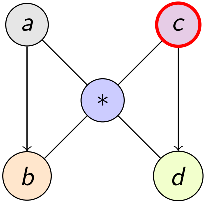

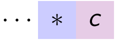

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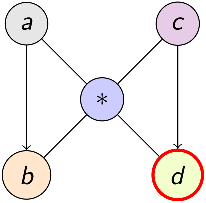

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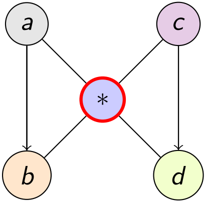

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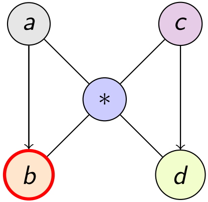

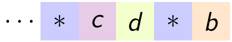

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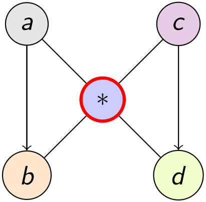

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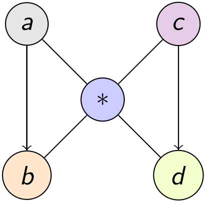

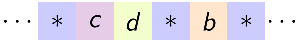

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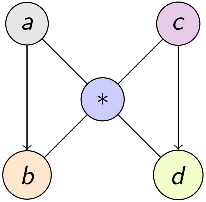

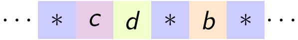

We call X_G the set of all the possible G-colourings of $\mathbb{Z}$.
Many properties of X_G can be decided by looking at the graph.

Hom shift

More generally, given a *non-directed* graph G, the Hom shift X_G in dimension d is the set of colourings of $\mathbb{Z}^d$:

- with the alphabet $V(G)$ the vertices of G
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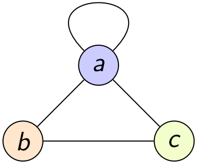

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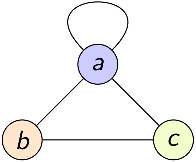
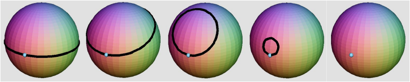


$\rightarrow$

c	a	b	c
b	a	a	b
a	b	a	a

Useful object from topology: the fundamental group of a space



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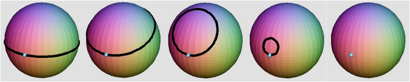

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Fundamental Group

Useful object from topology: the fundamental group of a space

Idea: trace paths in the space, and see how we can continuously deform them.
The fundamental group $\pi_1(X)$ is a topological invariant.

Fix some $x_0 \in X$.
We consider a particular set of paths, the loops based on x_0.
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$\pi_1(X, x_0)$ is the group of loops, concatenation being the group operation.

Some examples

Sphere:

$\pi_1 = \{e\}$, the trivial group.

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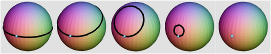

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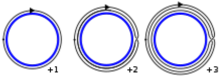

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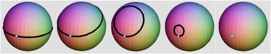

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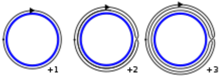

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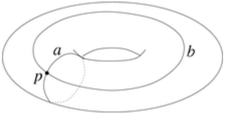

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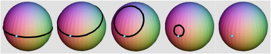

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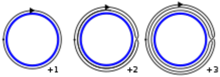

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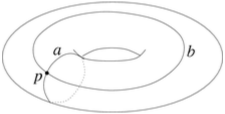

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Here, the fundamental group does not depend on the chosen basepoint x_0: those spaces are (path-)connected.

Fundamental group of a graph

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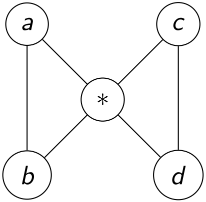

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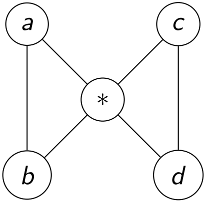

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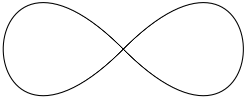
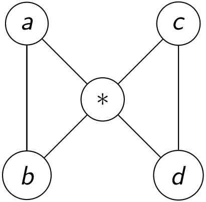


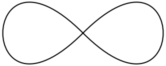
Topologically equivalent space

Fundamental group and spanning trees

How to compute the fundamental group of this "topological graph" ?



Graph G with m edges and n vertices



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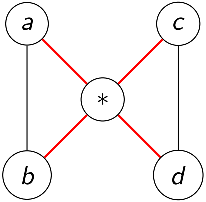

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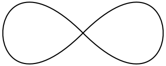

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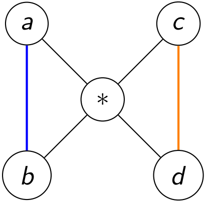

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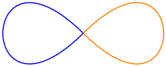

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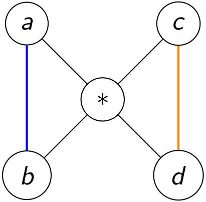

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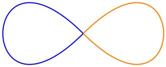

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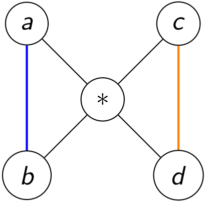
Intuitively: those edges are the "cycle edges"; each one corresponds to an non-contractible loop



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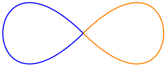

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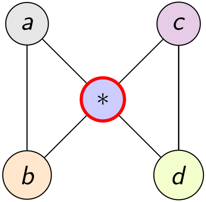
Idea: define "local" fundamental groups, and take the limit.

Loops in G and in X_G



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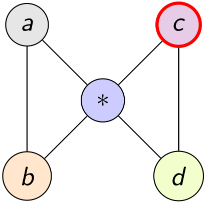

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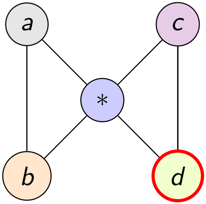

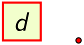

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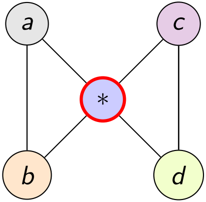

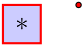

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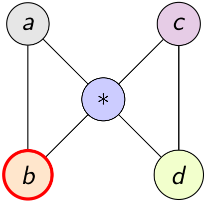

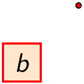

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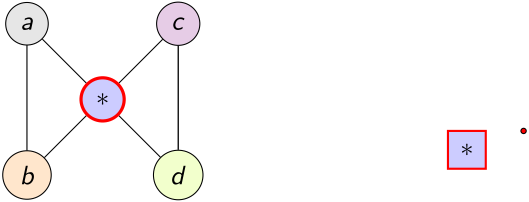

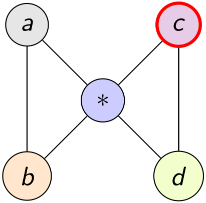
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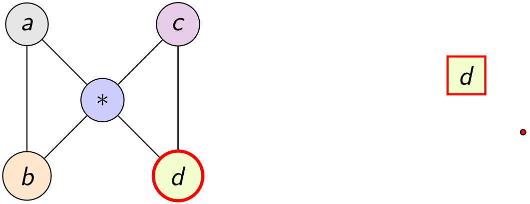

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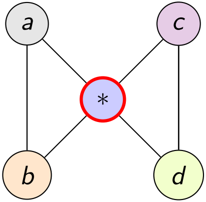

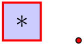

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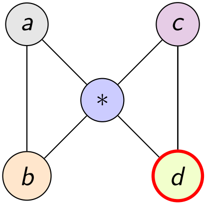

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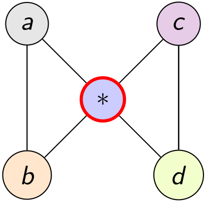

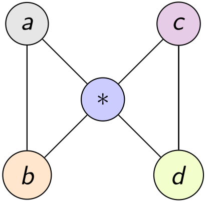

To a loop in the graph G, we can associate a sequence of (here 1×1) patterns of X_G + a closed loop in $\mathbb{Z}^2$.

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For a given "aperture window" $B \subset \mathbb{Z}^2$, a path will be:

- a *trajectory* path in $\mathbb{Z}^2$ (sequence of adjacent points)

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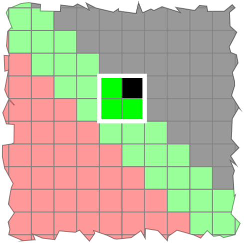

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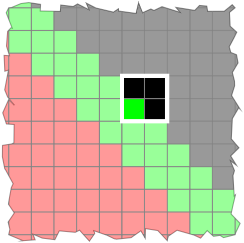

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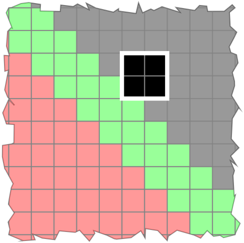

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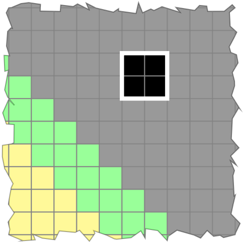

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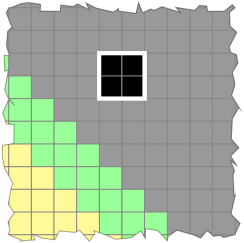

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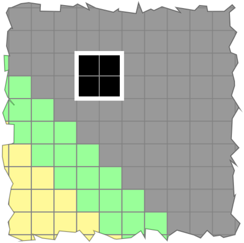

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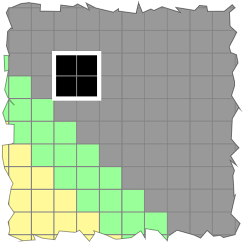

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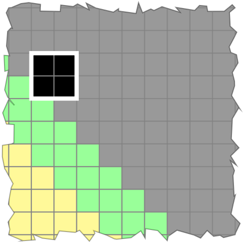

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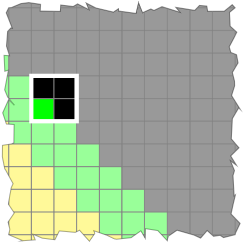

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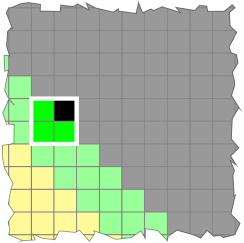

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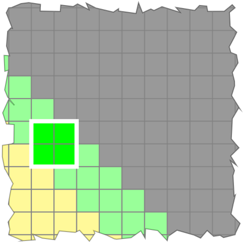

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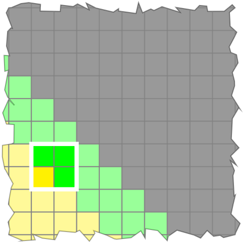

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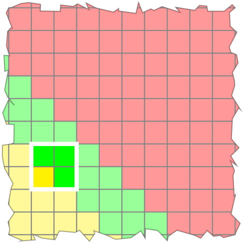

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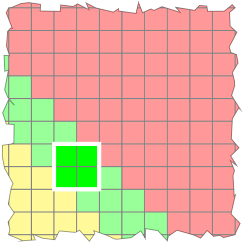

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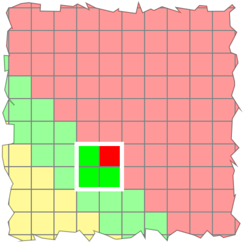

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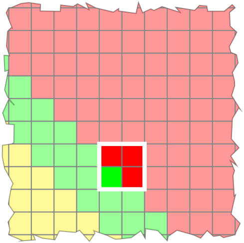

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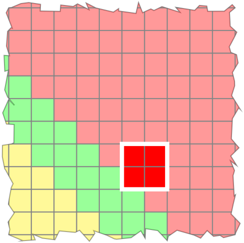

Intuitively, we can move inside a given configuration, or we can "jump" into another one.

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This gives us a notion of path. What about deformations ?

Any part of a path that can be traced inside a single configuration can be replaced with another trajectory *inside this configuration*.

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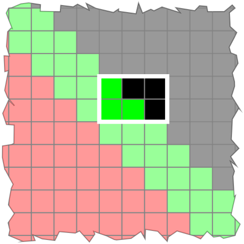

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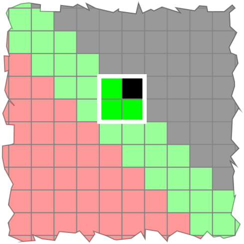
This path ... can be deformed into this one



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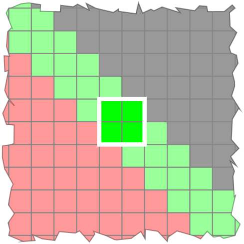

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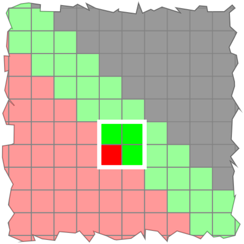

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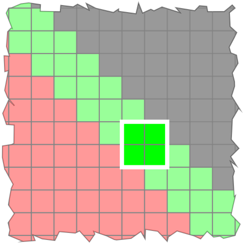

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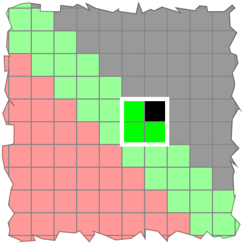

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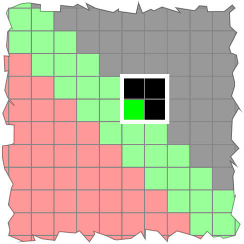

Homotopic paths: paths that be deformed into one another with a finite sequence of such elementary deformations.

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We can therefore define *for each window* $B \in \mathbb{Z}^2$ a fundamental group.

What about the projective part ?

This gives us one "fundamental group" *per aperture window*.

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Definition (Projective path)

Let $x, y \in X$. A projective path between x and y is a set of paths (p_n), with p_n having aperture window $B_n = \{-n, \ldots, n\}^2$, such that for all n:

- p_n is a path between the central patterns of support B_n of x and y
- p_{n+1} restricted to its central B_n window can be deformed in p_n

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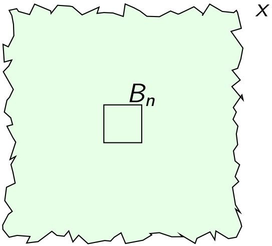

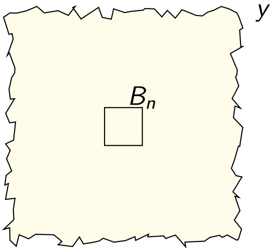

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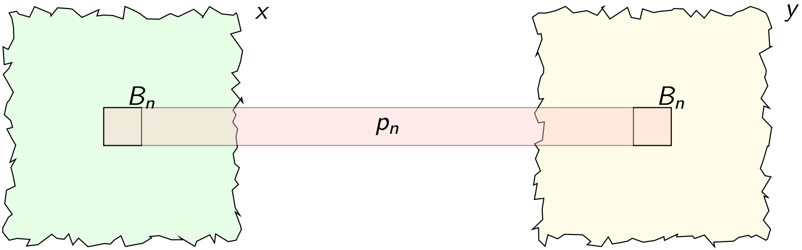

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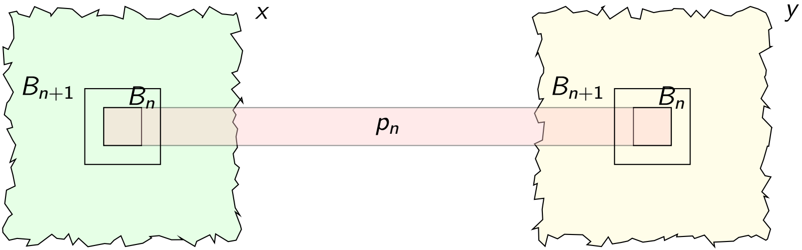

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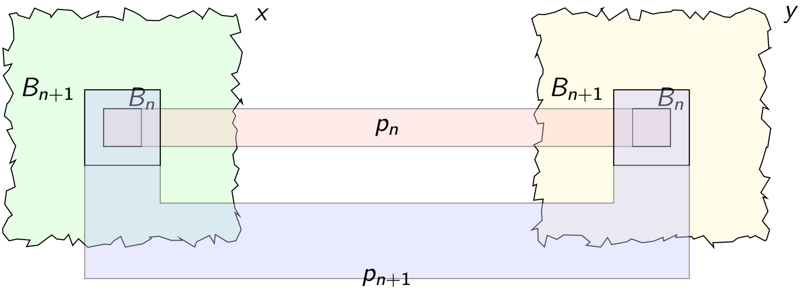

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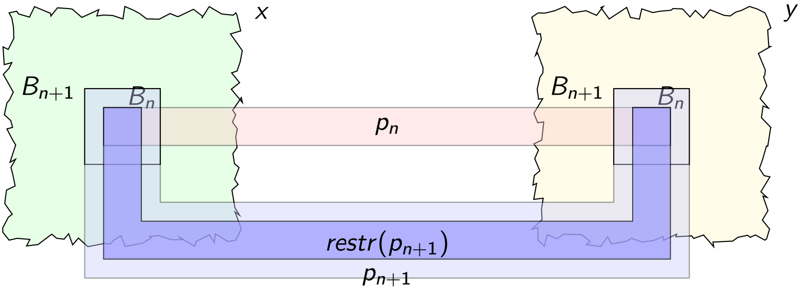

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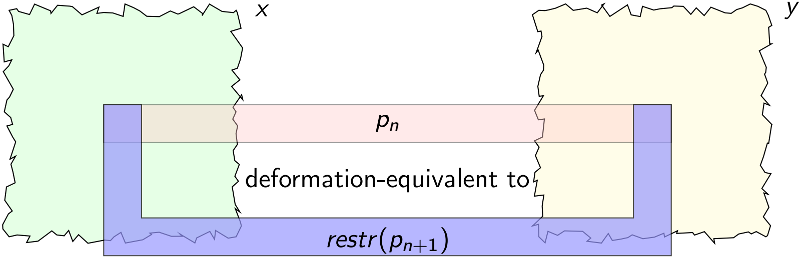

Trees

Theorem

Let T a – possibly infinite – tree. The projective fundamental group of X_T is trivial.

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Technical remarks:

- T is bipartite: the theorem is in fact about its two projective path-components.

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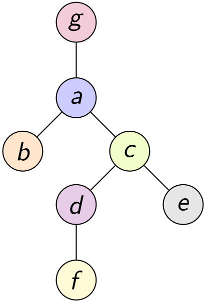
- T is bipartite: the theorem is in fact about its two projective path-components.
- the definition of the projective fundamental group does not actually require X to be a subshift, so T being infinite is OK.

Sketch of the proof: flip the deepest vertices



A tree T rooted in a

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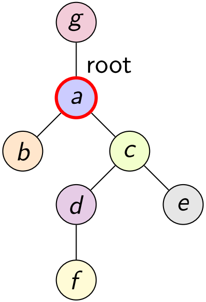
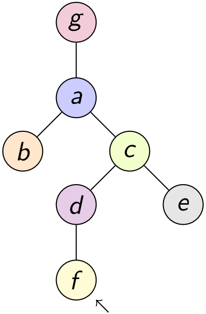


A tree T rooted in a

c	e	c	d	c	a
a	c	d	c	a	b
c	e	c	d	c	a
a	c	d	f	d	c
b	a	c	d	c	a
a	g	a	c	a	g

Configuration on X_T



Deepest vertex in T

c	e	c	d	c	a
a	c	d	c	a	b
c	e	c	d	c	a
a	c	d	f	d	c
b	a	c	d	c	a
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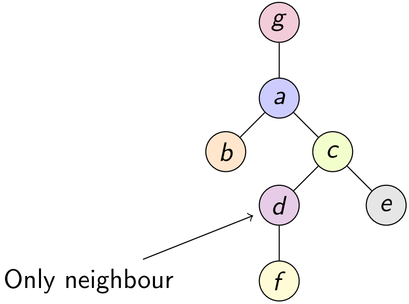

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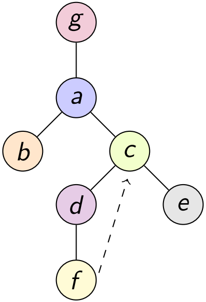

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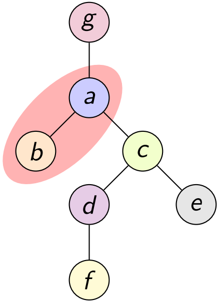

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b	a	b	a	b	a
a	b	a	b	a	b
b	a	b	a	b	a
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b	a	b	a	b	a
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Configuration on X_T

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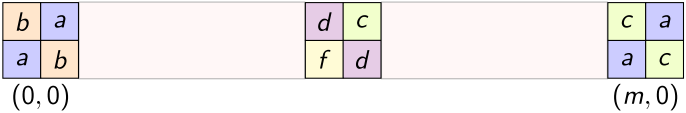

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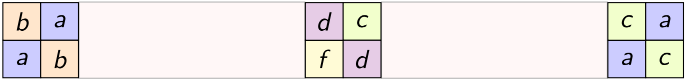

Extend the bottom side as before: go "up" in the tree

Sketch of the proof: deform in the chessboard configuration

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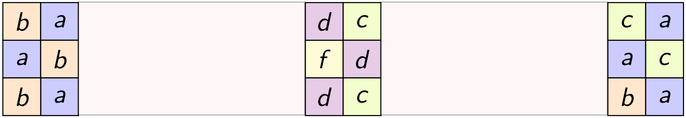

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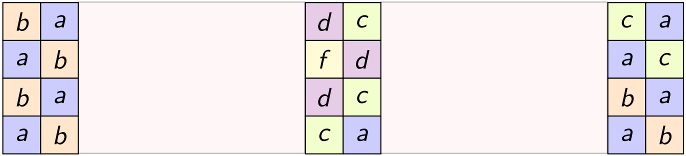

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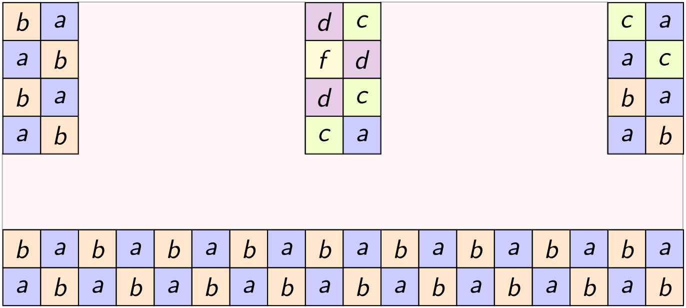

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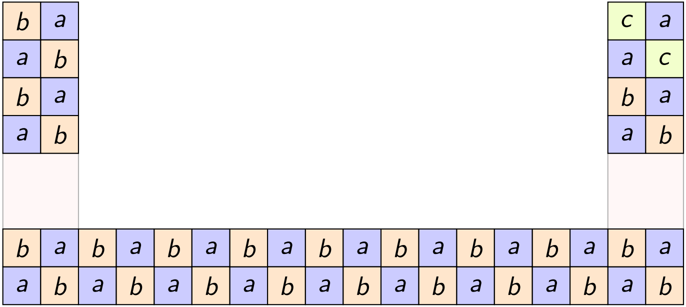

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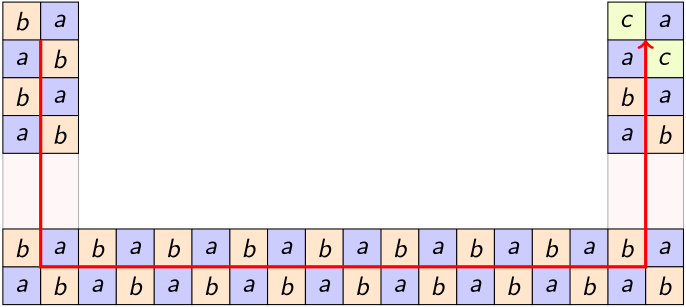

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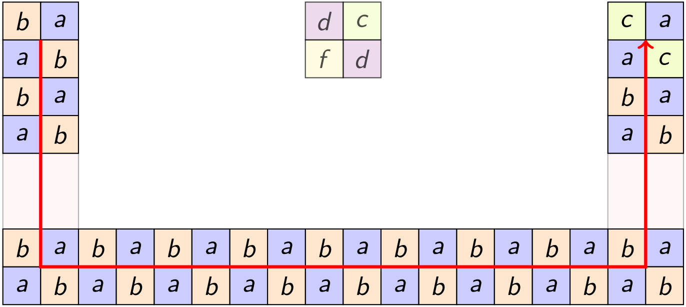

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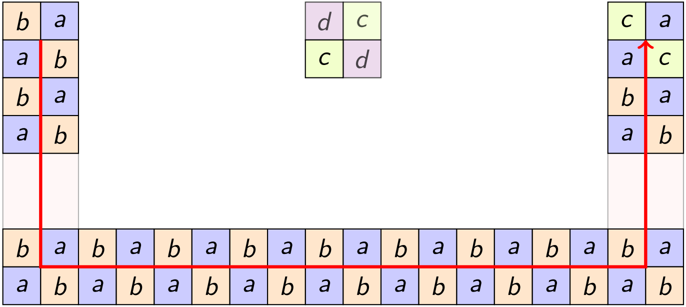

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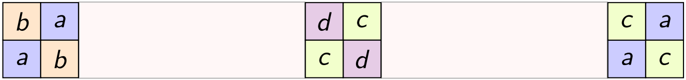

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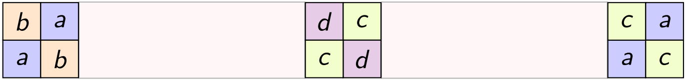

We either reduced the maximal "depth" reached in p_n or the number of times it is reached: repeat to get a contractible path in the chessboard.

Universal covering

An important idea in topology adapted here: *universal coverings*

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Definition (Universal Covering)

The universal covering of a graph G is the smallest tree $\mathcal{U}_G$ that admits a surjective morphism $\mathcal{U}_G \to G$

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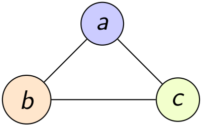

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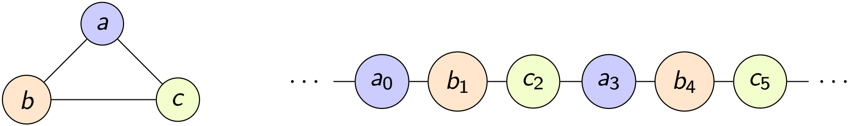

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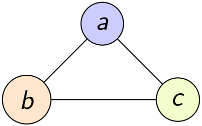

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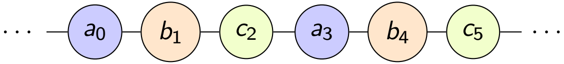

Universal cover of G

In the following, $\phi \colon \mathcal{U}_G \to G$ is one such surjective morphism.

Covering shift

Notation: For a graph G, we write $\hat{X}_G = X_{\mathcal{U}_G}$

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Note that if G a cycle, $\mathcal{U}_G$ is infinite: $\hat{X}_G$ is not *actually* a subshift (it is not compact, for example).

Lift

The covering $\phi \colon \mathcal{U}_G \to G$ induces a covering $\hat{\phi} \colon \hat{X}_G \to X_G$ (apply ϕ pointwise).

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c	a	b	c
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$\rightarrow$

c_2	a_3	b_4	c_5
b_1	c_2	a_3	b_4
a_0	b_1	c_2	a_3

Point of $\hat{X}_{C_3}$

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The "good condition" mentionned above: G contains no squares (cycles of length 4).

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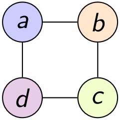

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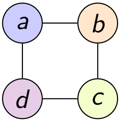

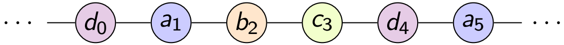

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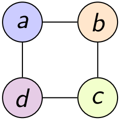

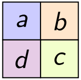

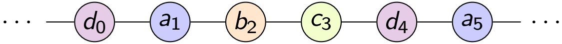

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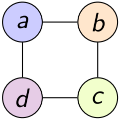
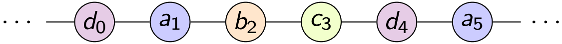


a	b
d	c

a_1	b_2
$d_?$	c_3



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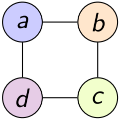

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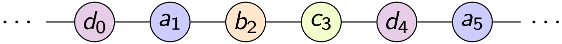

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Does not lift

The square-free case

This gives us the following theorem:

Theorem (Square-free)

Let G a non-bipartite, loop-free, square-free graph, with m edges and n vertices. Then, $\pi_1^{proj}(G) = F_{m-n+1}$, the free-group on $m-n+1$ generators.

Proof

Understand the link between

- the graph G and its universal covering $\mathcal{U}_G$

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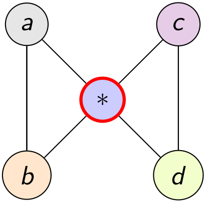
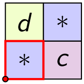
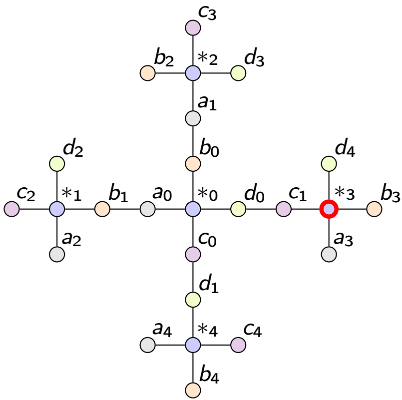
- and their respective fundamental groups $\pi_1^{proj}(X_G)$ and $\pi_1^{proj}(\hat{X}_G)$

A loop in X_G, in G and its lift in the universal covering







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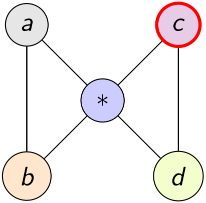

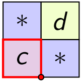

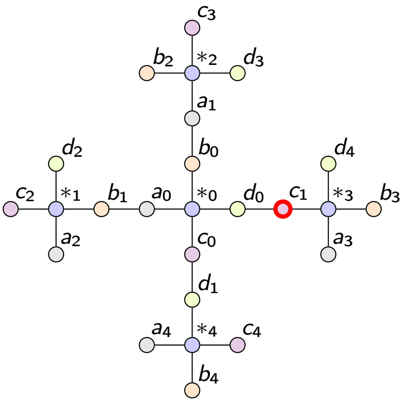

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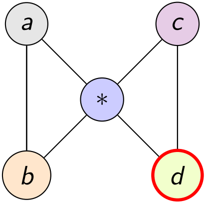

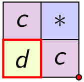

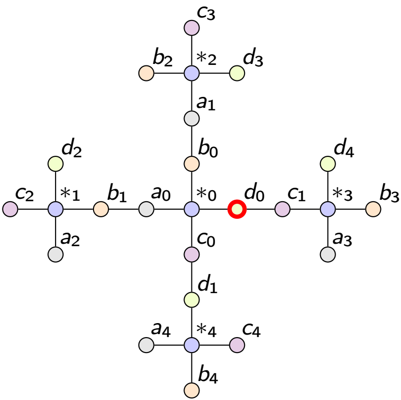

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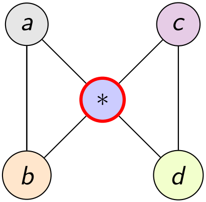

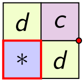

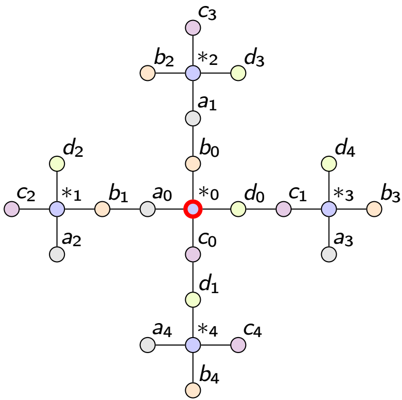

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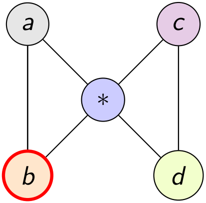

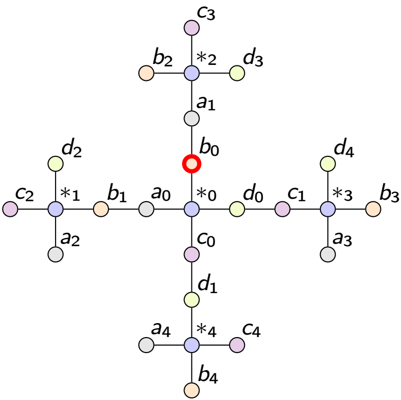

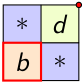

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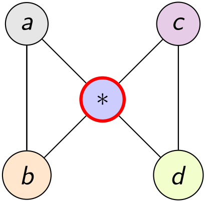

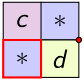

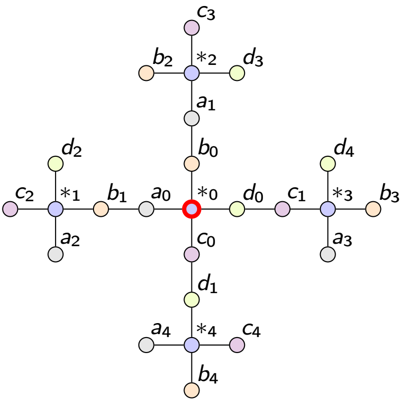

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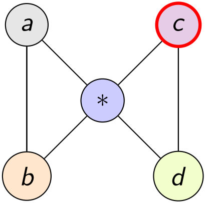

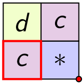

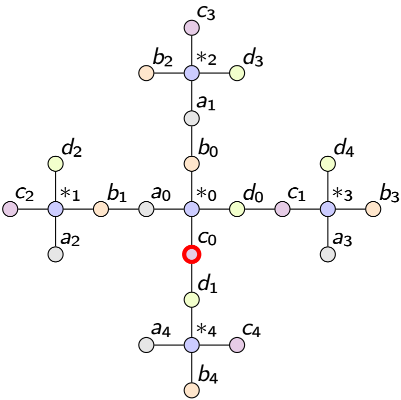

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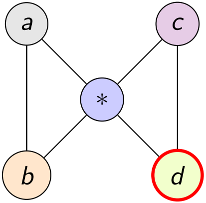

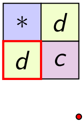

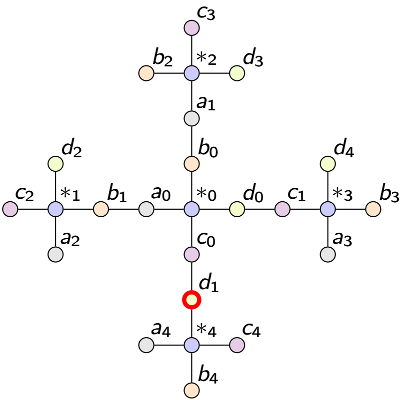

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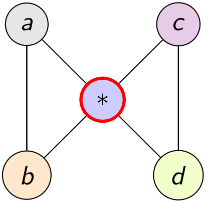

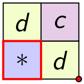

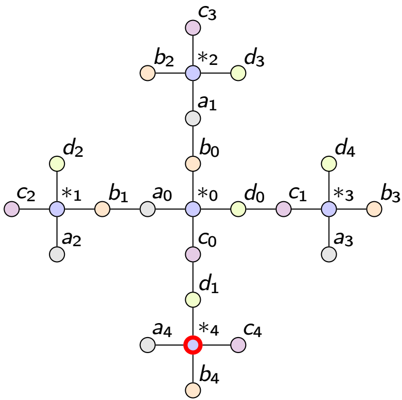

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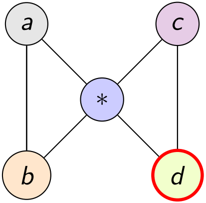

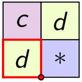

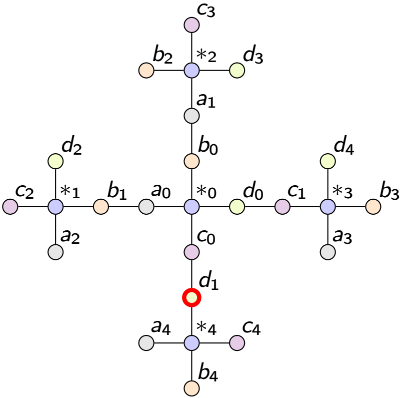

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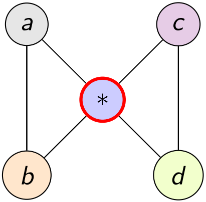

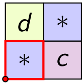

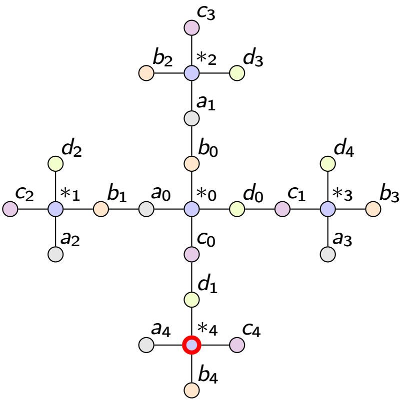

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- What about graphs containing squares ?
- In the general case: what is the complexity of computing the fundamental group ? Of deciding if the subshift is “projectively connected” ?

Bibliography

Nishant Chandgotia.
Four-cycle free graphs, height functions, the pivot property and entropy minimality.
Ergodic Theory and Dynamical Systems, 37(4):1102–1132, 2017.

William Geller and James Propp.
The projective fundamental group of a $\mathbb{Z}^2$-shift.
Ergodic Theory and Dynamical Systems, 15(6):1091–1118, 1995.