Topological Full Group and Tilings

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Alphabet: finite set of colours

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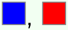
$\mathcal{A} = \{$  $\}$

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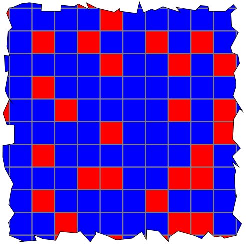

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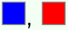
$\mathcal{A} = \{$  $\}$

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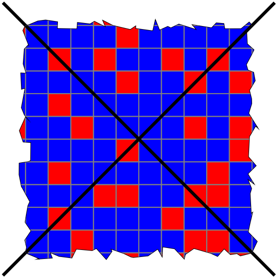

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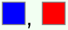
$\mathcal{A} = \{$  $\}$

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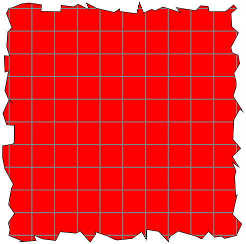

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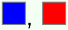
$\mathcal{A} = \{$  $\}$

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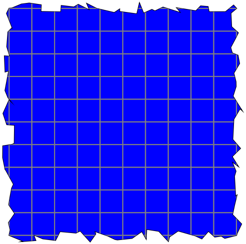

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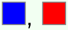
$\mathcal{A} = \{$  $\}$

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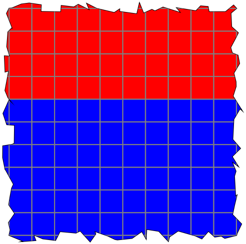

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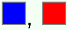
$\mathcal{A} = \{$  $\}$

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The set of all the valid configurations is called a **subshift**, denoted by $X_{\mathcal{F}}$

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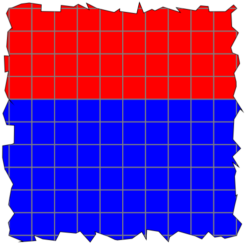

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Motivates a wide range of questions (extra-assumptions so that they become decidable ? Undecidable, but how much ? What kind of complicated objects can we obtain using only SFTs ?)

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A subshift is a closed set X verifying $\sigma^u(X) = X$ for all u (and so subshifts are compact).

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Remark: Continuous functions $X \to \mathbb{Z}^d$ depend only on a finite pattern around the origin of $\mathbb{Z}^d$.

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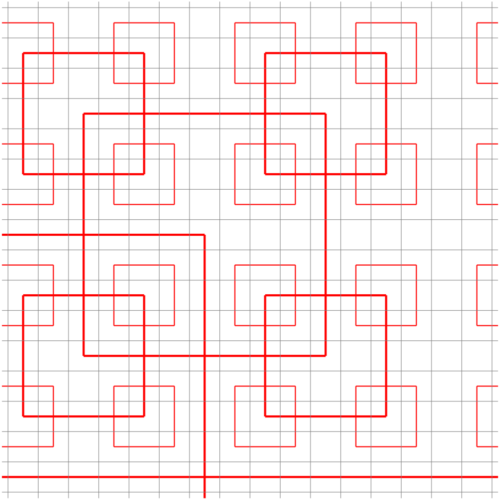
Formally, and with $\mathcal{L}_n(X)$ the patterns of support $[0, n-1]^d$:

$$\forall n > 0,\ \exists N \geq n, \forall\, w \in \mathcal{L}_n(X),\ \forall W \in \mathcal{L}_N(X), w \sqsubseteq W$$

Example of a minimal $\mathbb{Z}^2$-subshift



Topological full group: definition

Definition

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ a subshift. The **(topological) full group** of X is

$$[\![X]\!] = \{x \in X \mapsto \sigma^{\eta(x)}(x) \in \mathrm{Homeo}(X) \,|\, \eta \colon X \to \mathbb{Z}^d \text{ (continuous)}\}$$

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The full group of a subshift is a conjugacy invariant.

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Define for each colour $\blacksquare \in \{{\color{blue}\blacksquare}, {\color{red}\blacksquare}, {\color{green}\blacksquare}\}$ the involution:

$$\sigma_{\blacksquare}(x) = \begin{cases} \sigma(x) & \text{if } x_0 = \blacksquare \\ \sigma^{-1}(x) & \text{if } x_{-1} = \blacksquare \\ x & \text{otherwise} \end{cases}$$

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Claim: $\langle \sigma_{{\color{blue}\blacksquare}}, \sigma_{{\color{lime}\blacksquare}}, \sigma_{{\color{pink}\blacksquare}} \rangle = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 \leq [\![X_3]\!]$.

Sketch of proof

Let $\sigma_{c_{n-1}} \dots \sigma_{c_0} \in \langle \sigma_{\color{blue}\blacksquare}, \sigma_{\color{green}\blacksquare}, \sigma_{\color{red}\blacksquare} \rangle$. It acts non-trivially on $\cdots \cdot c_0 c_1 \dots c_n$ if $c_n \neq c_0$

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This is a way to construct infinitely many non-isomorphic simple, finitely generated, non-elementary amenable groups.

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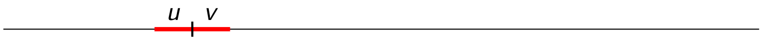

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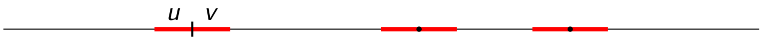

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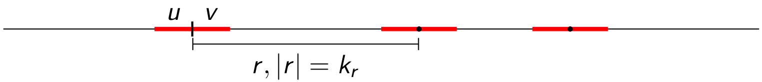

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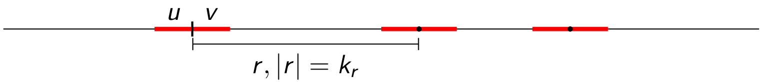

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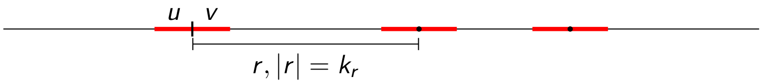

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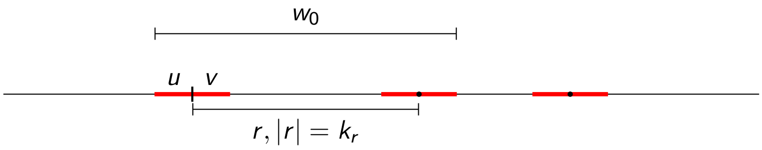

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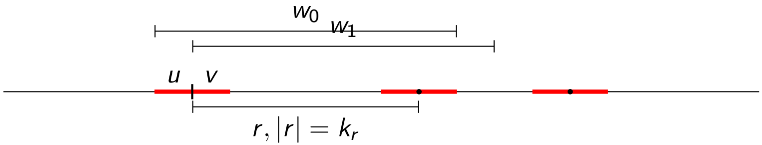

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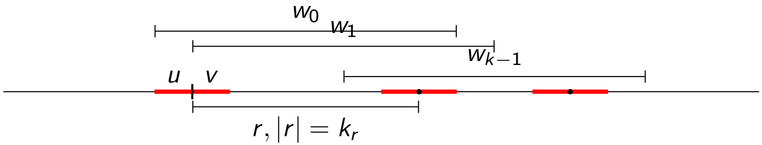

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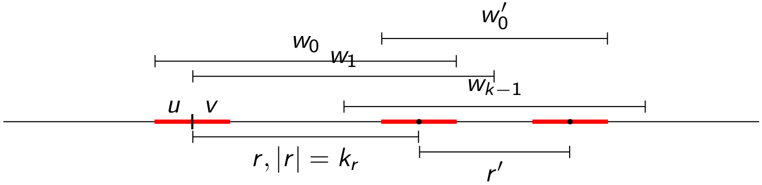

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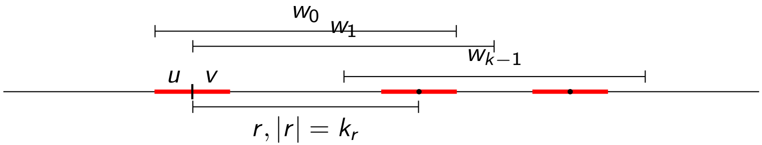
$\implies$ $\displaystyle\bigsqcup_{r \in R_n} \bigsqcup_{i<|r|} \sigma_i([u.rv])$ is a partition of X.



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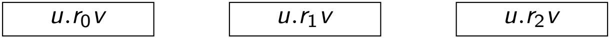

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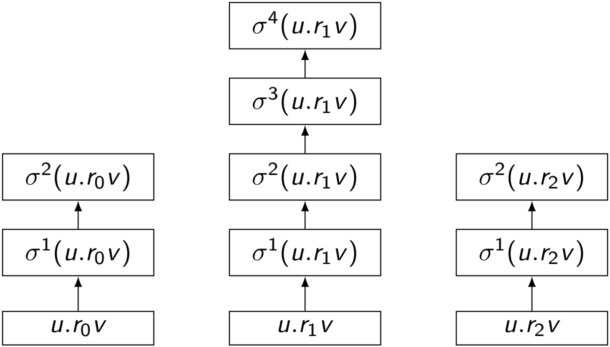

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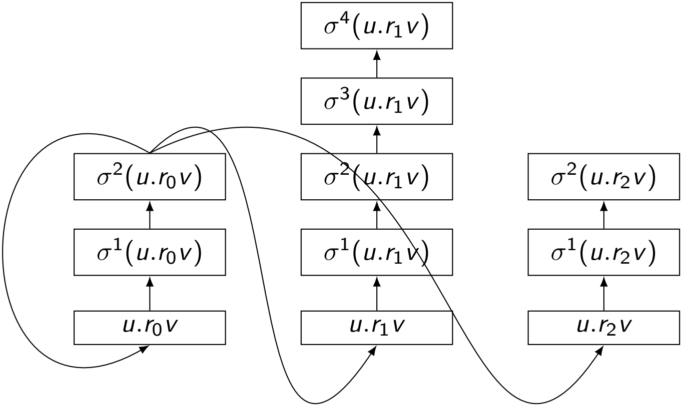

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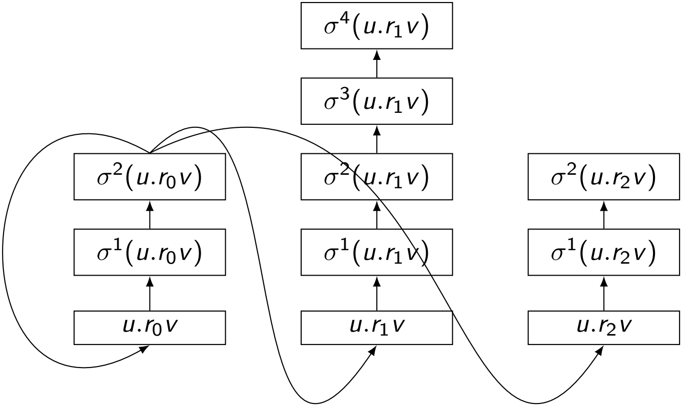

Within each tower, each shift only takes up one step higher, but at the top of the towers, we can fall back to *any* of the bases.

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Link with the full group:

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$$\prod_{r \in R(u.v)} \mathfrak{S}_{|r|} \leq [\![X]\!]$$

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$$[\![X]\!]_{x,n} = \prod_{r \in R(u_n.v_n)} \mathfrak{S}_{|r|}$$

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Note: the $([\![X]\!]_{x,n})_{n \in \mathbb{N}}$ are increasing.

Decomposition of the full group

Studying $[\![X]\!]_x \approx$ studying $\mathfrak{S}_n$. For example:

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Using this, and studying elements of $[\![X]\!]$ that "correspond" to 3-cycles, we can even compute an explicit presentation of $[\![X]\!]_x$ for minimal $\mathbb{Z}$-subshifts [GM18].

Things break in dimension 2

Construction from Elek and Monod ([EM13]): minimal $\mathbb{Z}^2$ subshift with non-amenable full group (impossible for $\mathbb{Z}$-subshifts)

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Idea: re-use the involutions defined previously (obtain a free subgroup), and "standard subshift tricks" to make the subshift recurrent – and so minimal.
Formally: we will consider some sub-subshift of the set of proper 6-*edge colourings* of $\mathbb{Z}^2$.

Elek-Monod construction

Alphabet: $\{-, -, \mathsf{I}, a, b, c\}$
Fix $(w_i)_{i\in\mathbb{N}} = (abb, ca, \dots)$ an enumeration of $\langle a, b, c \mid a^2 = b^2 = c^2 \rangle$.

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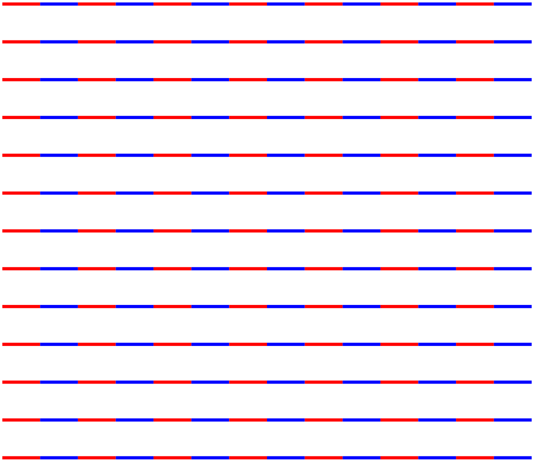

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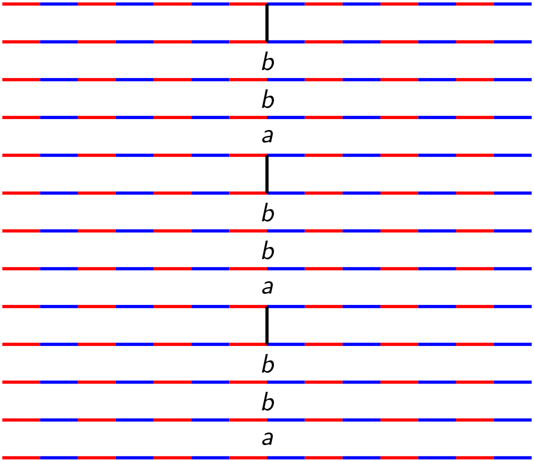

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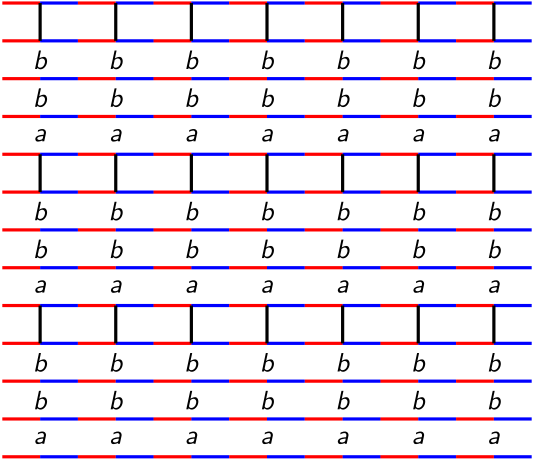

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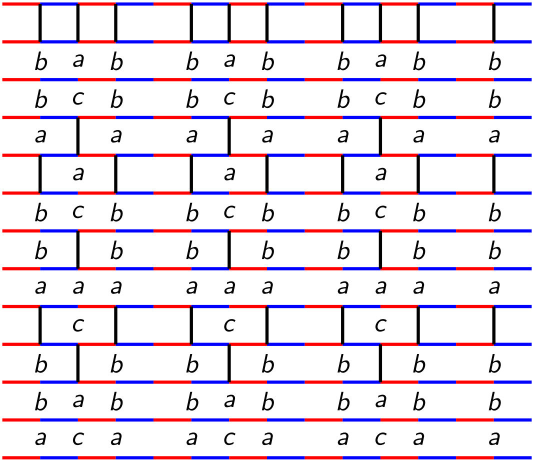

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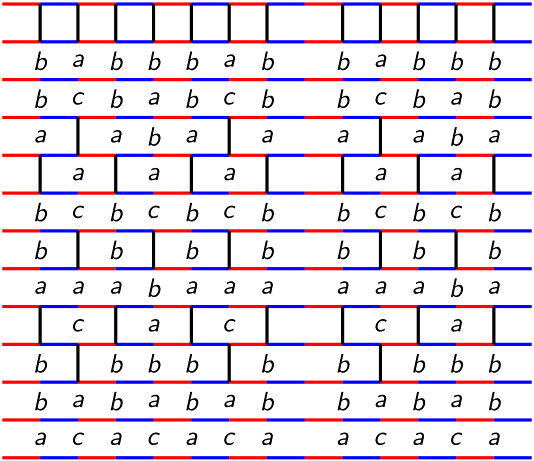

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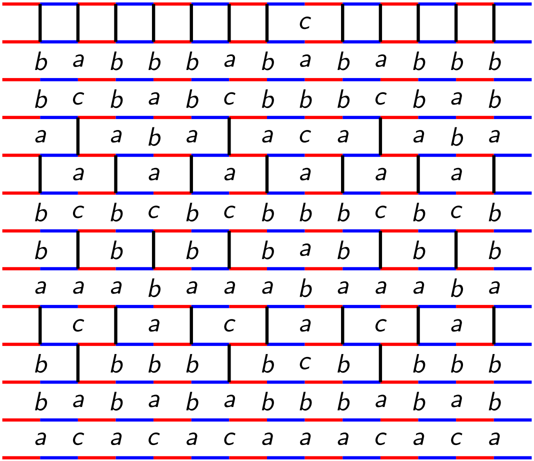

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In fact, the shift σ already acts freely on X ! Indeed, all the configurations are aperiodic.
For $\tau = \sigma_{c_{n-1}} \dots \sigma_{c_0} \in \langle \sigma_a, \sigma_b, \sigma_c \rangle$, there exists a configuration $x \in X$ whose column $\{0\} \times \mathbb{N}$ is $(c_0 \dots c_{n-1} \mathsf{I})^\infty$, and $c_0 \neq \mathsf{I}$ so $\tau(x) \neq x$.

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The previous example show that we lose amenability in higher dimensions. Do we have something more ?

Lamplighters

We define the *d-dimensional lamplighter group* as

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\begin{aligned}
L_d &= \mathbb{Z}_2 \wr \mathbb{Z}^d \\
&= \left\langle a, s_1, \ldots, s_d \mid a^2, (awaw^{-1})^2 \text{ for all } w \in \{s_1, \ldots, s_d\}^*, s_i s_j s_i^{-1} s_j^{-1} \right\rangle
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- For $0 \leq i \leq d$, s_i move one step in direction $e_i \in \mathbb{Z}^d$ (so the s_i commute)

Embed L_d in $[\![X]\!]$?

Proposition ([BB22])

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This "dimension" obstruction does not hold for $\mathbb{Z}^2$-subshift !

Embed L_2 in $[\![X]\!]$ a bidimensional subshift

Take X the sunny-side-up: configurations on $\{\square, \text{[yellow square]}\}$ with at most one [yellow square].

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Idea: encode the state of the lamplighter in the "parity" of the $\blacksquare$. The action is clearly not free, but this is still enough to have
$\langle \sigma_h, \sigma_v, \tau \rangle = L_2 \leq [\![X]\!]$

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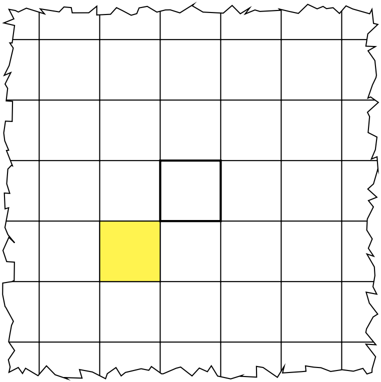

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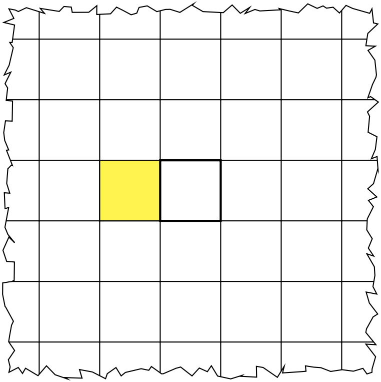

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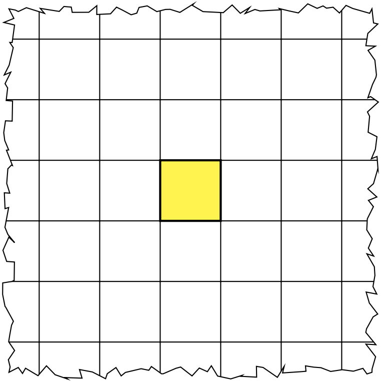

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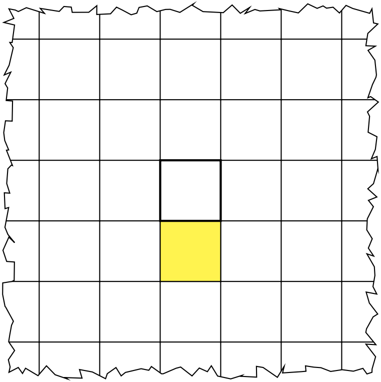

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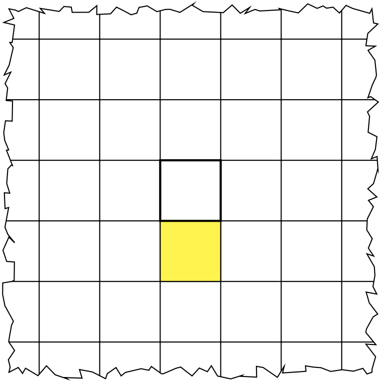

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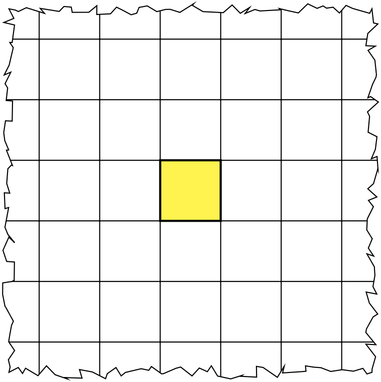

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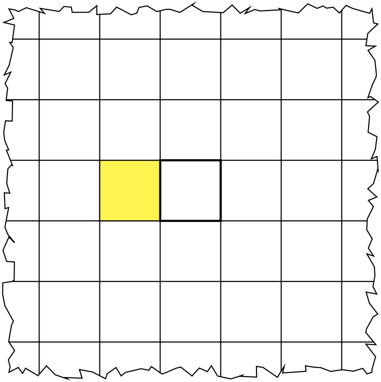

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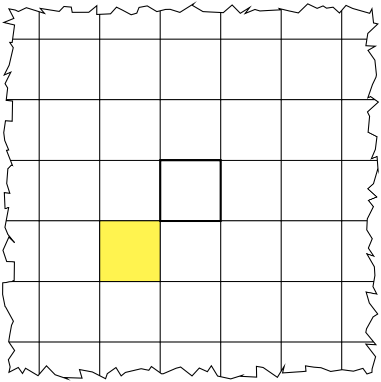

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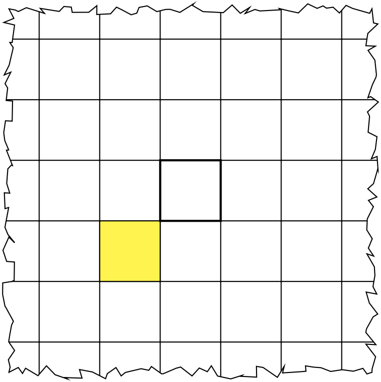

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Some of those algebraic properties heavily depend:

- On the *structure* of the subshift (minimal or not, recursive language and decidable word problem ...)

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Some of those algebraic properties heavily depend:

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Some natural questions: how complex can TFGs of multidimensional SFT be (*e.g.* how hard are their word or torsion problems) ? Are there are other obstructions to embeddability than growth rates of group actions ?

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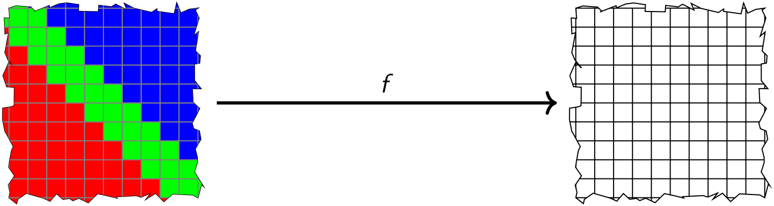

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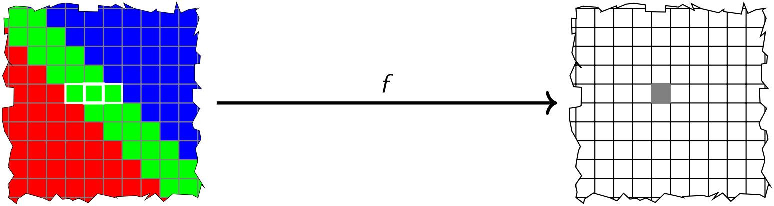

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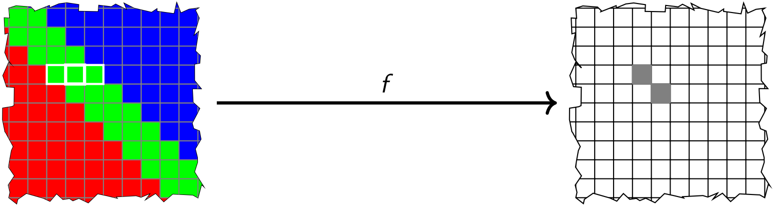

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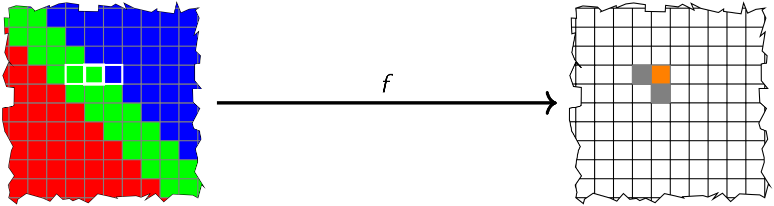

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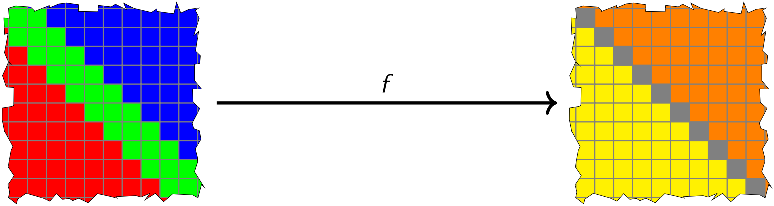

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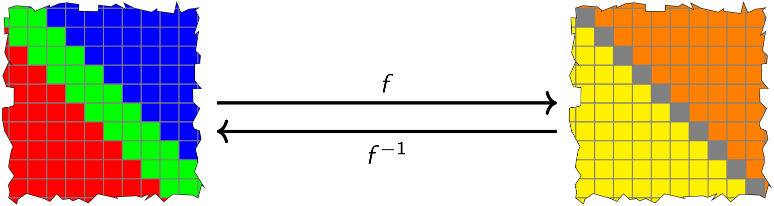

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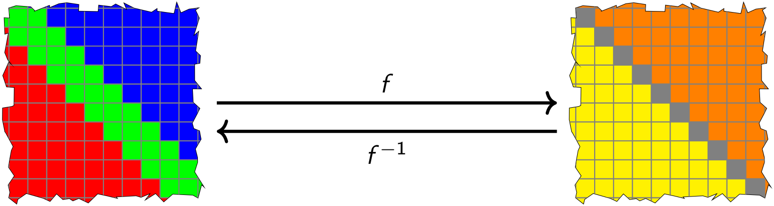

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"Properties" preserved by conjugacy are called *conjugacy invariants*

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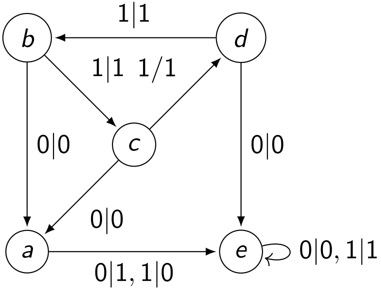

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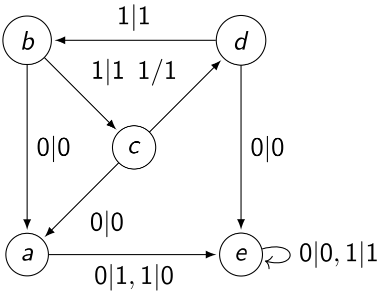

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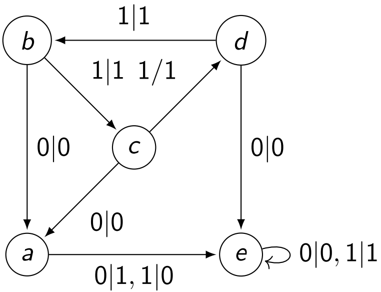

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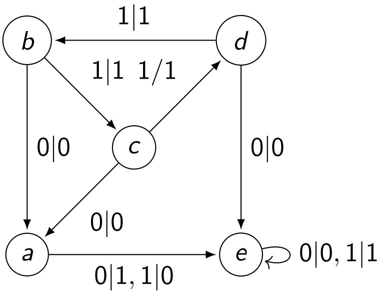
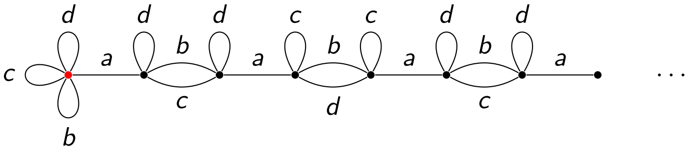


Notation $p|q$: when reading p, move to the correct state and output q.
Starting from state c and reading 1100 $\implies$ end in state e, output 1101.
Each state induces an automorphism of $\{0,1\}^{\mathbb{N}}$, and $\mathcal{G} = \langle a, b, c, d \rangle$.

Embed $\mathcal{G}$ in some full group

Previous definition makes it clear that $\mathcal{G}$ acts on $\{0,1\}^{\mathbb{N}}$, and the graph of the action looks like this:



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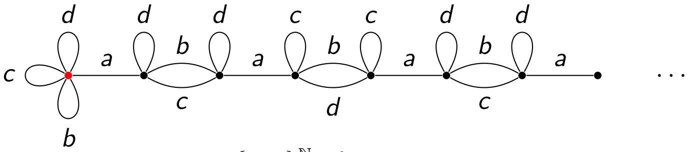

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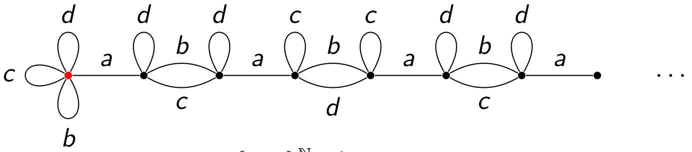

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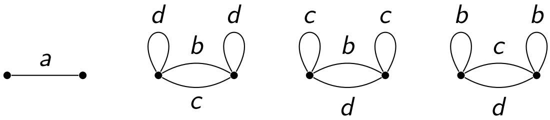
More or less a linear shape: in fact, we can obtain it as a tiling of $\mathbb{Z}$ by the graphs



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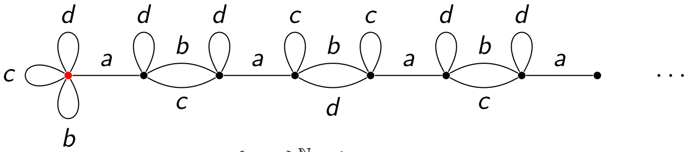

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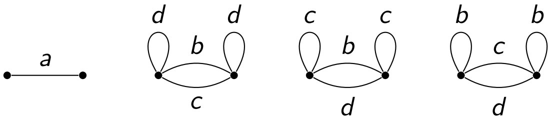

Using the usual involutions σ_i which follow the edge $i \in \{a,b,c,d\}$ in both directions at the origin, we have $\mathcal{G} \in [\![X]\!]$.