Realizing Finitely Presented Groups as Fundamental Groups of SFTs

Léo Paviet Salomon, Pascal Vanier

GREYC
Université Caen-Normandie

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Tilings

Alphabet: finite set of colours

$\Sigma = \{$ ,  $\}$

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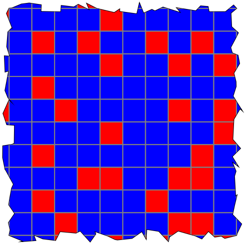
$$\mathcal{F} = \emptyset$$

Configuration on the alphabet Σ $\iff$ Σ-colouring of $\mathbb{Z}^2$ that does not contain any forbidden pattern



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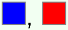
$\Sigma = \{$  $\}$

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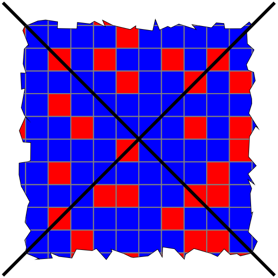

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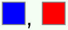
$\Sigma = \{$  $\}$

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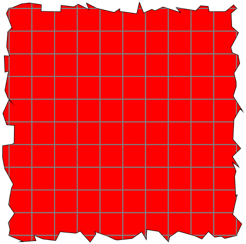

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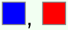
$\Sigma = \{$  $\}$

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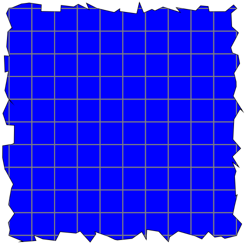

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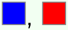
$\Sigma = \{$  $\}$

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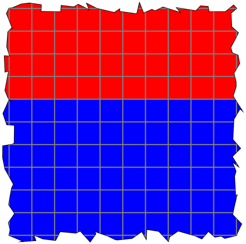

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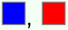
$\Sigma = \{$  $\}$

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The set of all the valid configurations is called a **subshift**, denoted by $X_{\mathcal{F}}$

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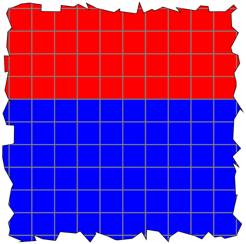

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- Is X empty ?
- Does X contain an aperiodic configuration ?
- How many patterns of size $n \times n$ are there in X ?

Projective Fundamental Group

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Idea: mimic the definition of the fundamental group in topology.

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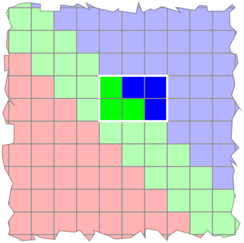

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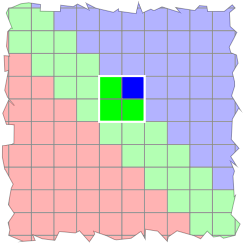

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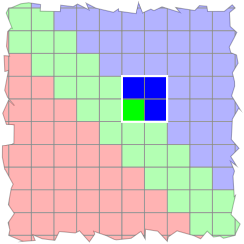

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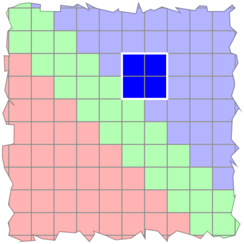

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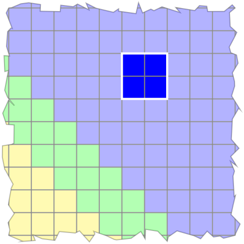

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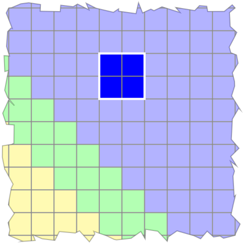

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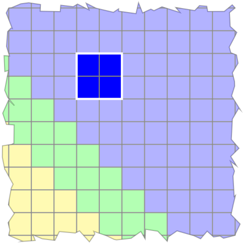

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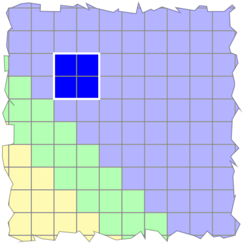

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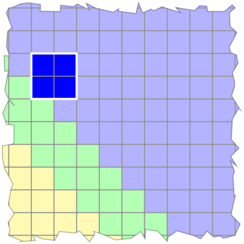

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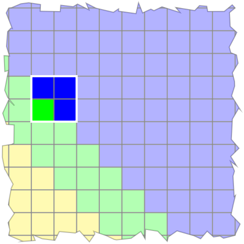

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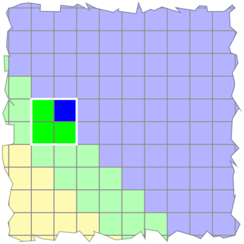

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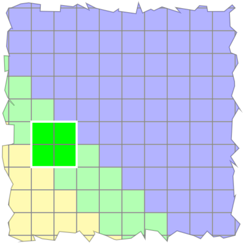

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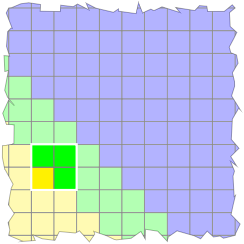

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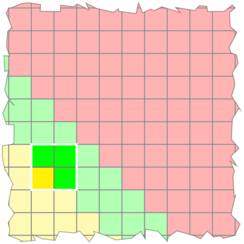

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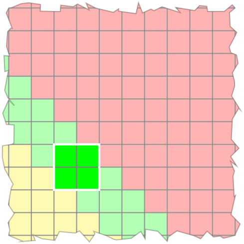

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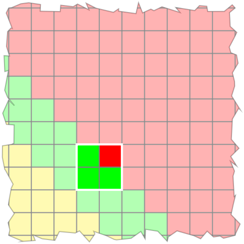

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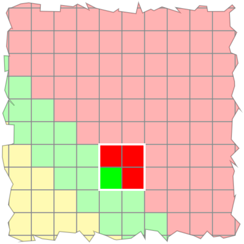

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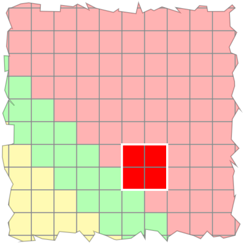

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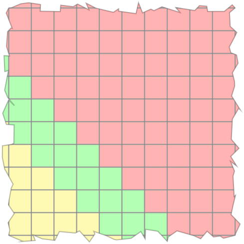

A path that stays within a single configuration is **coherent**.

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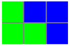

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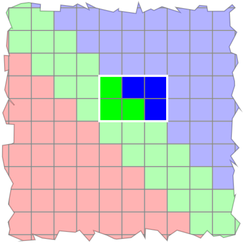

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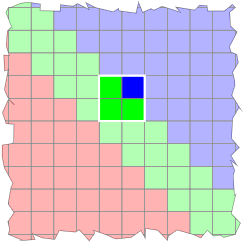

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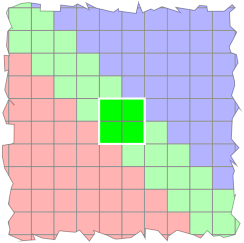

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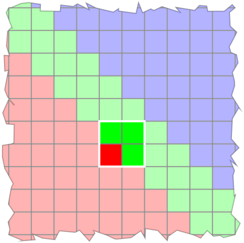

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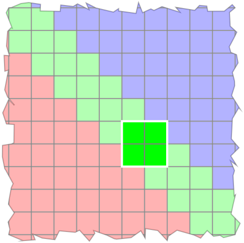

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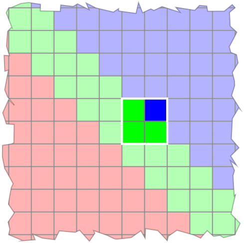

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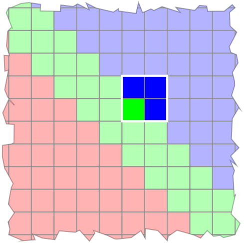

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(To get a single object, we need a way to combine all those groups $\pi_1^B(X)$ in a single group $\pi_1(X)$: we take the projective limit of those groups – not important here)

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A first answer in the original article: we can at least make all the **finite groups**.
We show that we can in fact realize (at least) all the **finitely presented** groups.

Group presentation: example

Point of view about group presentations: **rewriting theory**

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$$\begin{aligned} abbab &= abb\color{red}{bb'}\color{black}ab \\ &\underset{r_2}{=} ab'ab \\ &\underset{r_3}{=} ab'a\color{red}{aba'b'}\color{black}b \\ &\underset{r_1,r_2}{=} ab'ba' \\ &= aa' \\ &= 1 \end{aligned}$$

Main theorem

Theorem (P., Vanier [PSV23])

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The idea of the construction is to see paths in the subshift as words on the alphabet of generators, and their deformation as relations. More precisely:

- Some tiles will correspond to generators
- Paths will correspond to words on those generators
- Local rules will ensure that deformations correspond to insertion/deletion of relators

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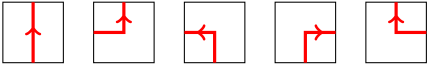
For each generator $s \in S$:

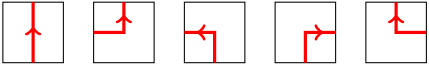


and a single empty tile

A set of tiles

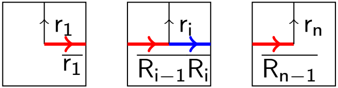
Let $G = \langle S \mid R \rangle$ a finitely presented group. Consider the tiles:

For each generator $s \in S$:



and a single empty tile

For each $r_1 \dots r_n \in R$:



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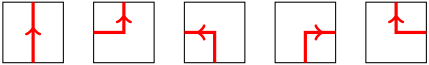

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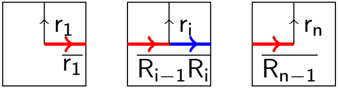

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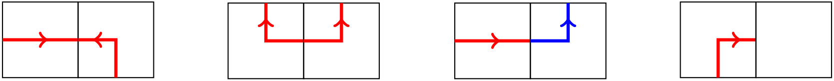
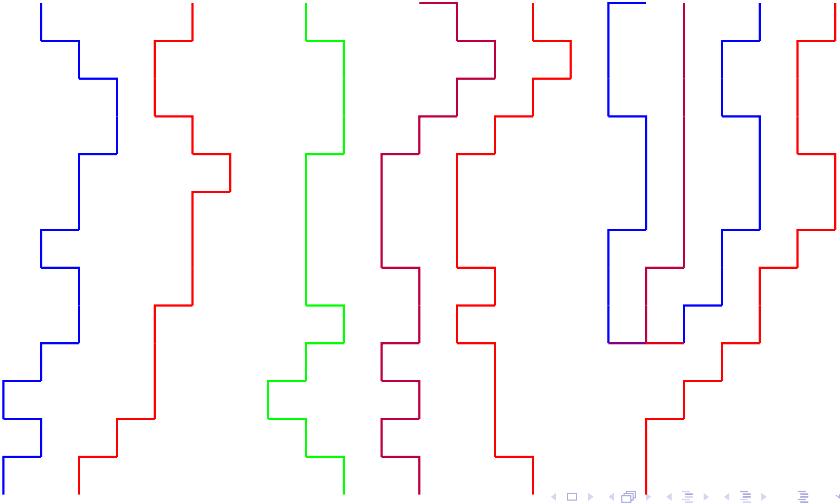
Matching rules: wires can't stop, *i.e.* all the following are **forbidden**



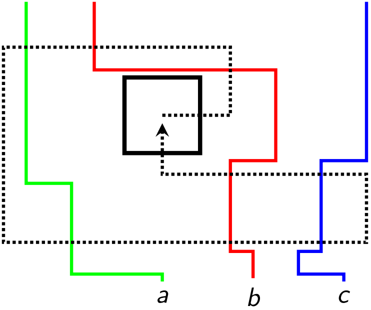
Configurations

Configurations look like this:



From paths to words

To a path p, we associate a unique word on the alphabet $S \cup S^{-1}$, denoted by $[\![p]\!]$, defined by the wires that it crosses.



For this path p, $[\![p]\!] \equiv bb^{-1}a^{-1}abcc^{-1}b^{-1} =_G 1_G$.

Homotopic implies equal

Claim: in a given configuration and between 2 given endpoints, all the paths are associated to the same element of G

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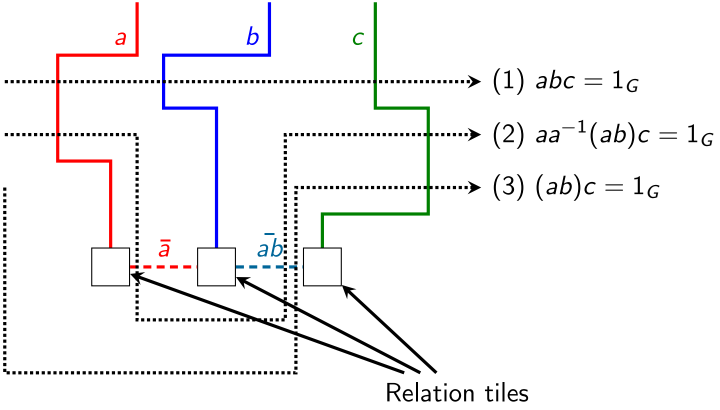
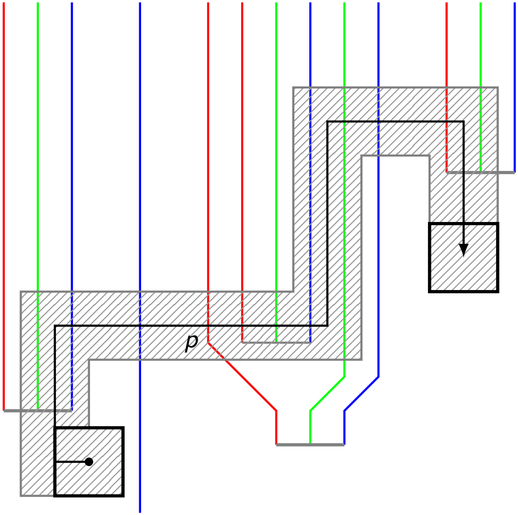


Relation $abc = 1_G$

Avoid all the relation tiles



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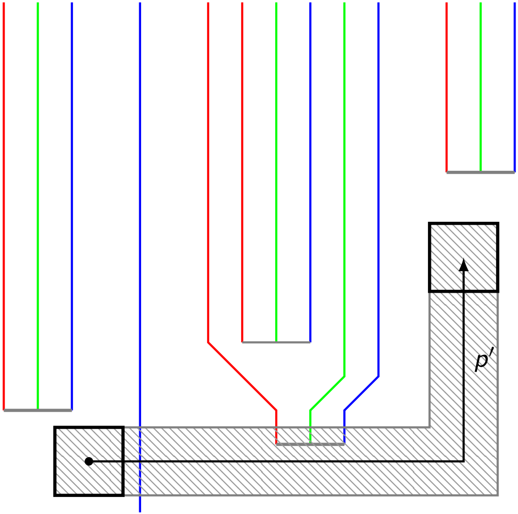

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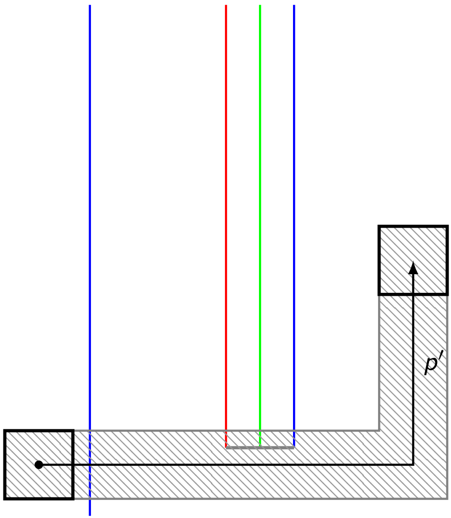

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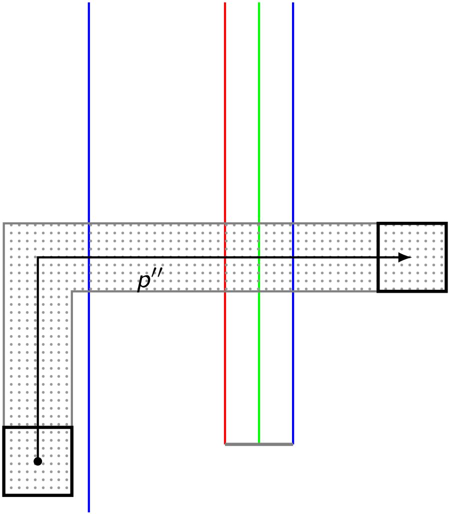

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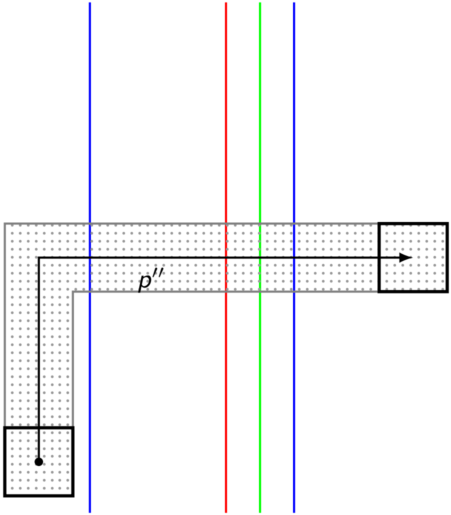
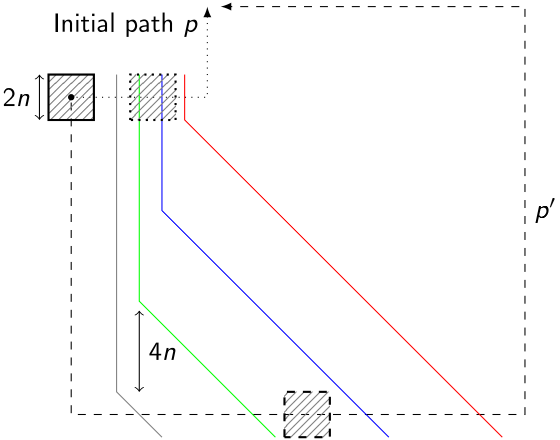


Only a single wire at a time

Deformation of p into p' in a single configuration to see only one wire per pattern.



Series of such tricks:

- All the equivalent loops are associated to the same group element
- All the loops associated to the same group element are equivalent

Therefore the fundamental group is $G = \langle S \mid R \rangle$

Bibliography

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Normalizing coherent paths

The previous remarks mean that a coherent path can be put into some kind of normal form. However, all loops need not be coherent – even when the fundamental group is trivial !

Normalizing coherent paths

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Solution: decompose a loop into a sequence of coherent paths, and add contractible loops at the junction.

Path decomposition

Whenever we have a incoherent path, we can split it in *normalized* coherent paths:

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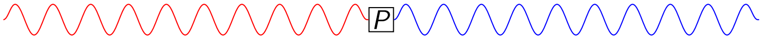

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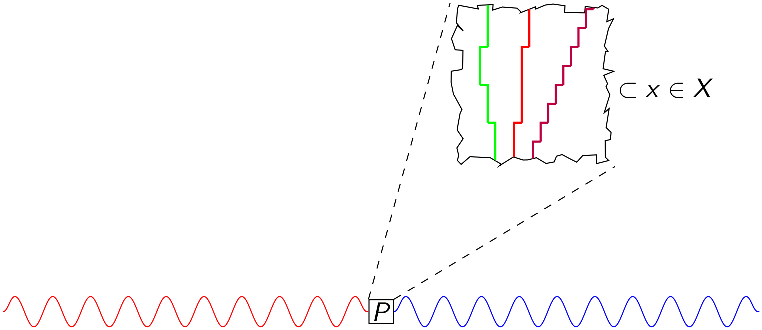

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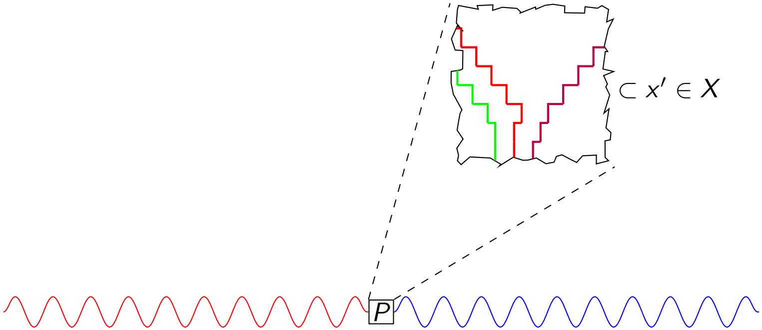

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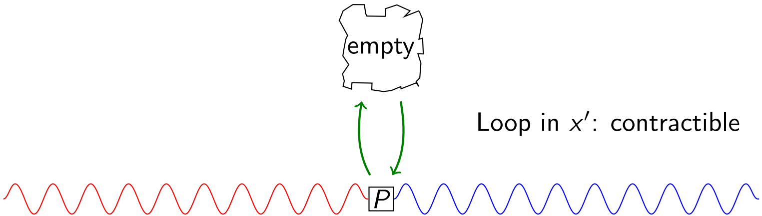

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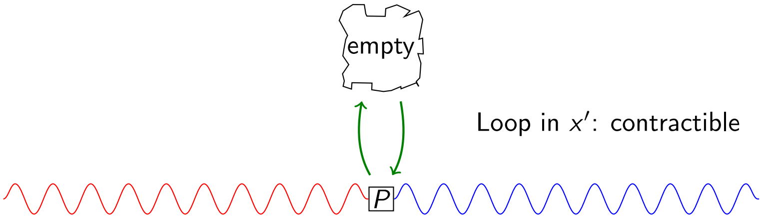

This means that we can deform the path so that each sub-coherent path starts and ends with empty patterns: can deform them independently.