Computability of extender sets in multidimensional subshifts

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Subshifts

Alphabet: finite set of colours

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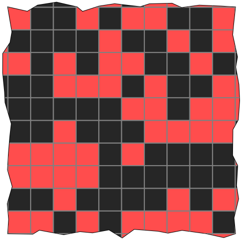

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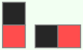
$\mathcal{F} = \{$  $\}$

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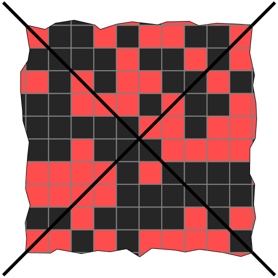

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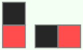
$\mathcal{F} = \{$  $\}$

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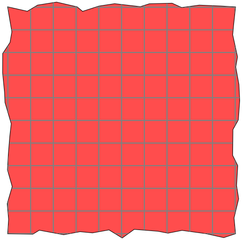

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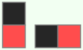
$\mathcal{F} = \{$  $\}$

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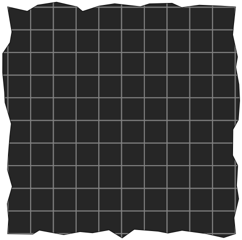

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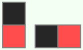
$\mathcal{F} = \{$  $\}$

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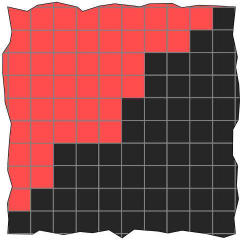

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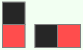
$\mathcal{F} = \{$  $\}$

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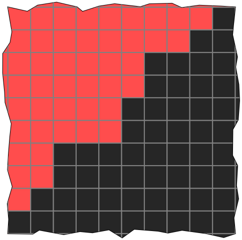

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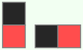
$\mathcal{F} = \{$  $\}$

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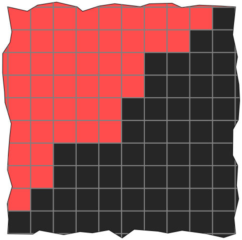

Classes of subshifts: if $\mathcal{F}$ is ...

- finite: **subshift of finite type**
- regular (d=1): **sofic**
- recursively enumerable: **effective**

Sofic subshifts: general definition

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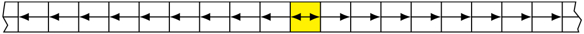

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This quantifies how many classes of patterns can be freely exchanged in **any** configuration: patterns are equivalent if they can appear in exactly the same "contexts" in X.

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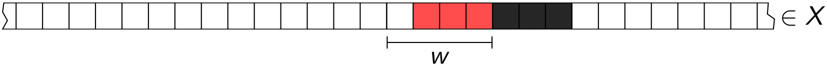

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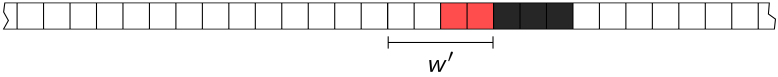

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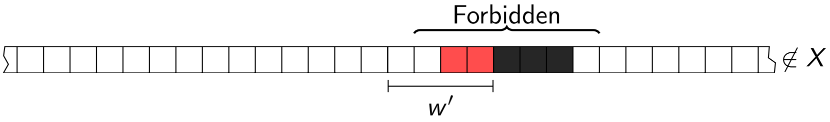

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Theorem (Ormes and Pavlov, 2016)

If X is a $\mathbb{Z}^d$-subshift, and if there exists some $n > 0$ such that $E_X(n) \leq n$, then X is sofic.

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Very slow growth rate of E_X is sufficient to be sofic.
Goal: understand the possible behaviours of the function E_X (for X sofic or effective, on $\mathbb{Z}$ or $\mathbb{Z}^2$, etc).

Extender entropy

Let X any subshift in any dimension d, over some alphabet $\mathcal{A}$. We define its **extender entropy** as

$$h_E(X) = \lim_{n \to +\infty} \frac{\log E_X(n)}{n^d} = \inf_{n \to +\infty} \frac{\log E_X(n)}{n^d}$$

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More informally: (up to some recoding of X as a binary subshift) $h_E(X) = \alpha$ means that there is only $0 \leq \alpha \leq 1$ bit of information per cell.

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In our particular problem, what we really need is the notion of **computable real numbers**.

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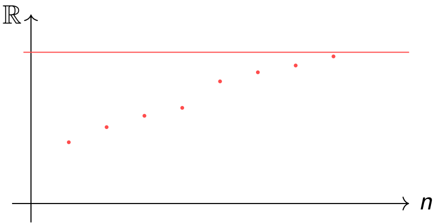

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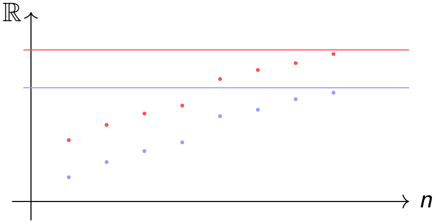

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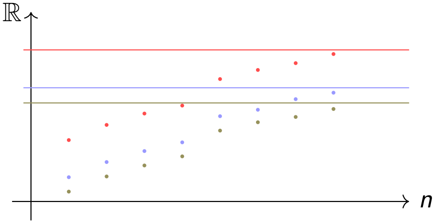

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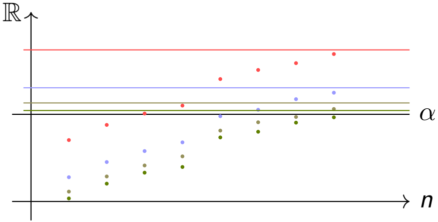

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Characterization: effective $\mathbb{Z}$ subshifts

We prove the following theorem:

Theorem (Callard, Vanier, P., 2023+)

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"Easy" direction: $h_E(X) = \inf_n \langle \text{something} \rangle$ is always a Π_3 real number.
Other direction: we construct for any $\alpha \in \Pi_3$ an effective $\mathbb{Z}$-subshift X_α, with $h_E(X_\alpha) = \alpha$.

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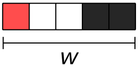

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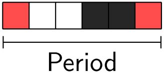

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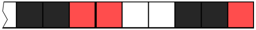

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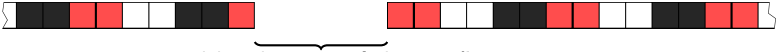

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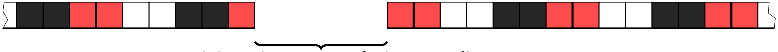

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If all the configurations of X are periodic, we can relate $E_X(n)$ to the number of n-periodic configurations.

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- Arbitrarily large periods + compactness: X contains non-periodic points, how do we deal with those ?

Characterization: sofic $\mathbb{Z}^2$ subshifts

Theorem (Callard, Vanier, P., 2023+)

The set of extender entropies of sofic $\mathbb{Z}^2$-subshifts is exactly the non-negative Π_3 real numbers.

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The set of extender entropies of sofic $\mathbb{Z}^2$-subshifts is exactly the non-negative Π_3 real numbers.

The fact it not always 0 has already been remarked in Destombes and Romashchenko, 2022 (setting: necessary conditions to be sofic in terms of resource-bounded Kolmogorov complexity).

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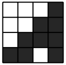

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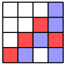

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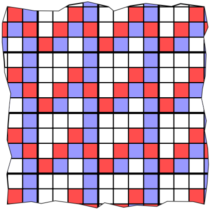

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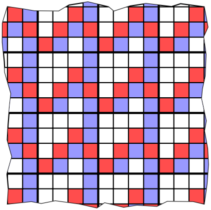

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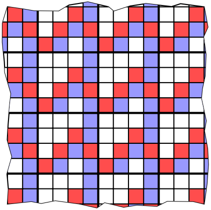

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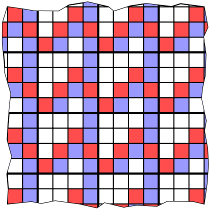

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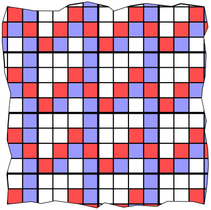

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- Main tool to construct "complicated" $\mathbb{Z}^2$ uses effective $\mathbb{Z}$ subshift with the right properties: we cannot really impose *bi*dimensional properties directly.
- Idea: set the density using standard theorems (Aubrun-Sablik, Durand-Romashchenko-Shen), and only have a single bit be periodic rather than entire blocks.

Summary

Extender entropies:

<table>
<tr><th></th><th>$\mathbb{Z}$</th><th>$\mathbb{Z}^{d\geq 2}$</th></tr>
<tr><td>SFT</td><td colspan="2">$\{0\}$</td></tr>
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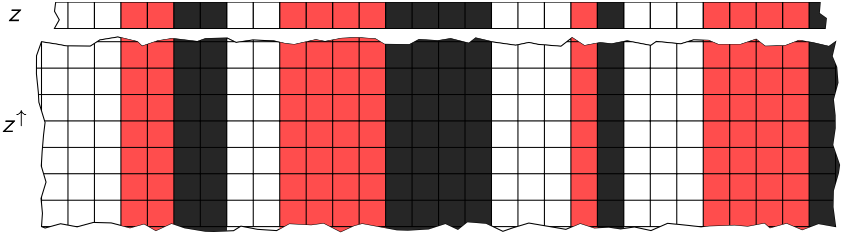

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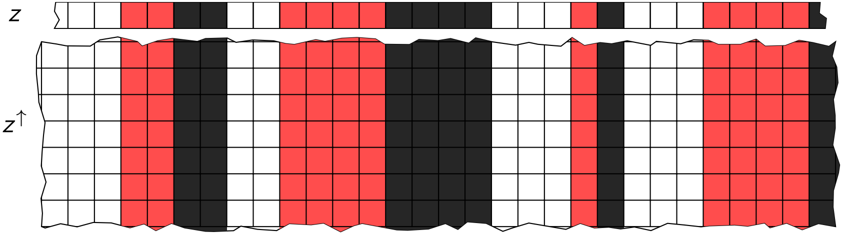

A very important theorem (used here as a “black-box”):

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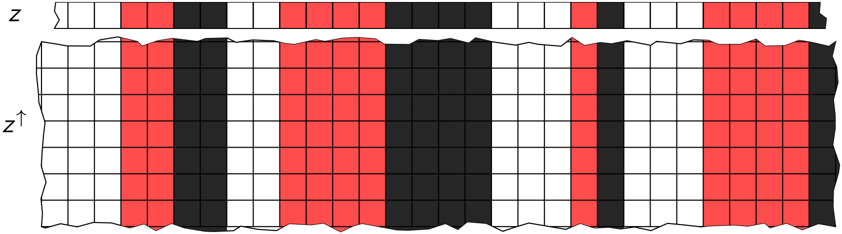

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Allows us to re-use the $\mathbb{Z}$-effective construction for sofic $\mathbb{Z}^2$ subshifts.

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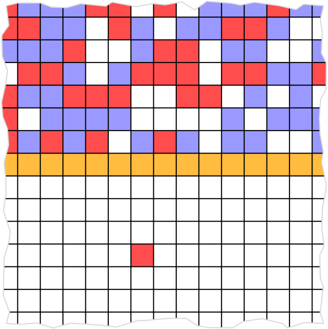

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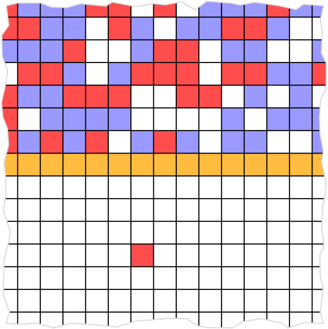
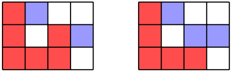


As before, different patterns have different extender sets.



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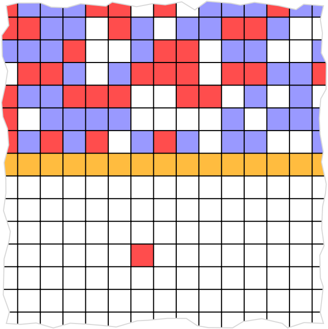

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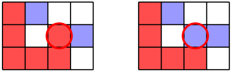

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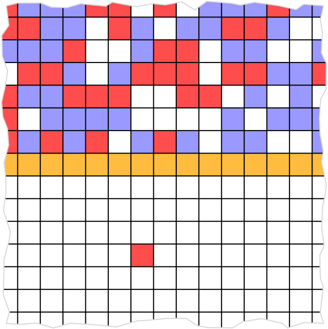

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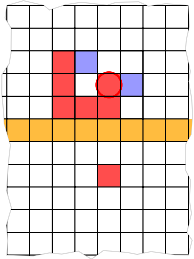

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