Realizing finitely presented groups as projective fundamental groups of SFTs

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Abstract

Subshifts are sets of colourings – or tilings – of the plane, defined by local constraints. Historically introduced as discretizations of continuous dynamical systems, they are also heavily related to computability theory. In this article, we study a conjugacy invariant for subshifts, known as the projective fundamental group. It is defined via paths inside and between configurations. We show that any finitely presented group can be realized as a projective fundamental group of some SFT.

2012 ACM Subject Classification Theory of computation → Models of computation; Mathematics of computing → Discrete mathematics

Keywords and phrases Subshifts, Wang tiles, Dynamical Systems, Computability, Subshift of Finite Type, Fundamental Group

Funding This research was partially funded by ANR JCJC 2019 19-CE48-0007-01

Acknowledgements The authors would like to thank all the anonymous referees of this paper for their remarks which helped improve the exposition.

1 Introduction

A d-dimensional subshift is a set of colourings of $\mathbb{Z}^d$ by a finite number of colours which avoid some family of forbidden patterns. If the family is finite, it is called a subshift of finite type (SFT). Most problems concerning subshifts in dimension $d \geq 2$ are undecidable [6, 20, 19], due to the fact that sets of Wang tilings are SFTs.

Together with the shift action σ, a subshift forms a dynamical system. Interesting dynamical aspects are usually invariant by conjugacy, which is the isomorphism notion for subshifts. Most conjugacy invariants of subshifts in dimensions $d \geq 2$ are linked to computability theory or complexity theory. Historically, the first example was the characterization of the topological entropies of multi-dimensional SFTs as the upper semi-computable numbers [25]. Afterwards, many other computational characterizations of conjugacy invariants have been obtained: growth-type invariants [30], subactions [23, 4, 14] and so on.

Links between groups and subshifts have recently seen a surge in interest with several different approaches: subshifts can be defined on groups instead of $\mathbb{Z}^d$ [1, 3] and some properties of the group are linked to decidability questions on the subshifts on it [26, 2, 15]. Analogies between groups and subshifts have allowed new characterizations to be proved for subshifts [27].

Another avenue is to associate a group to a subshift in order to construct conjugacy invariants in several ways [29, 22, 17]. The most well-known such group is the automorphism group, which is still not very well understood: for instance, while it is known that SFTs with positive entropy have very complex automorphism groups [24] or that SFTs whose automorphism group has undecidable word problem can be constructed [18], it is still not known whether the automorphism groups of the full shifts on 2 and 3 symbols are the same. Apart from the low complexity setting [13, 12] not much is understood about it.

In this article, we study another group-related conjugacy invariant called the *projective fundamental group* introduced by Geller and Propp [17]. Fundamental groups are an object of interest in several fields of theoretical computer science, in particular graph reconfigurations [37], which bear links with a particular class of subshifts called *hom-shifts* [10] which are defined with a graph of allowed adjacency of colours. These are subshifts with a computable language that still exhibit interesting behavior [16]. An essential tool in their study is their *universal cover*, a graph which has strong ties to their projective fundamental group. Fundamental groups are also of interest when studying the “defects” in tilings [33, 5], or obstruction to the tileability of finite, untiled “holes” in tilings [11, 36]. In particular, provided that an SFT satisfies some mixing-like hypothesis, there is an explicit link between its fundamental cocycles [35, 34] and its projective fundamental group.

In the usual topological setting (see for example [21]), the fundamental group $\pi_1(X)$ of a space X is a topological invariant which describes the number of holes and the general shape of X. It is defined as the group of equivalence classes of loops through continuous deformation, together with the composition operation. In this setting, the fundamental group is well-defined only when X is path-connected.

When viewed as subspaces of the Cantor space, subshifts are totally disconnected. Nevertheless, one can still define a notion of projective fundamental group using paths and deformations (see Subsection 3.1 for details). As in the classical setting, this notion is only well-defined in the case of *projectively connected subshifts*, the appropriate notion of path-connectedness. This property resembles mixing properties (see for instance [9] or [32]), but it is not known whether any of the mixing properties defined in [9] imply projective connectedness of an SFT, although some partial results exist [33, 35]. Projective connectedness is undecidable but we do not know how hard: it is open whether it belongs to the arithmetical hierarchy.

As a conjugacy invariant, the fundamental group allows one to distinguish between some subshifts which share the same entropy and periodicity data. It is also better understood than the automorphism group in the sense that the authors in [17] explicitly compute it for several well-known subshifts: the full shifts on any alphabet always have trivial fundamental group, the square-ice has $\mathbb{Z}$ and k-to-1 factors of full shifts – *i.e.* in which every point has exactly k preimages by the factor map – always have a fundamental group with finite order k. They also prove that any group of finite order is realizable as a fundamental group of some SFT.

The main result of this article is that any finitely presented group can be the fundamental group of an SFT:

▶ **Theorem 1.** *Let $G = \langle S|R\rangle$ be a finitely presented group. Then, there is a subshift of finite type X satisfying:*

- *X is projectively connected,*
- *the projective fundamental group of X is isomorphic to G.*

We do not think that this constitutes a characterization of projective fundamental groups of SFTs, as we do not have a matching upper bound on the hardness its word problem. However, this theorem implies that the hardness of the word problem of the fundamental group – *i.e.* given a SFT, decide the word problem of its fundamental group – can be any recursively enumerable degree [8], and in particular that its upper bound is at least Σ^0_1-hard [31, 7]. It also implies that any undecidable property on finitely presented groups is undecidable for projective fundamental groups.

The main construction of the paper is quite different from other constructions used in undecidability results on tilings and subshifts: it does not use an aperiodic subshift.

The paper is organized as follows. After recalling the symbolic dynamics background in Section 2, we introduce the projective fundamental group in Subsection 3.1, some examples in Subsection 3.2 and finally in Section 4 we prove Theorem 1.

2 Definitions

A **d-dimensional full shift** on some finite alphabet Σ is the set $\Sigma^{\mathbb{Z}^d}$, together with the **shift-actions** $\sigma_{\mathbf{u}}\colon \Sigma^{\mathbb{Z}^d} \to \Sigma^{\mathbb{Z}^d}$ defined for $\mathbf{u} \in \mathbb{Z}^d$ by $\sigma_{\mathbf{u}}(x)(\mathbf{v}) = x(\mathbf{u}+\mathbf{v}) = x_{\mathbf{u}+\mathbf{v}}$. The underlying topology is the one induced by the **Cantor distance**, defined on $\Sigma^{\mathbb{Z}^d}$ by

$$d(x,y) = 2^{-\min\{\|\mathbf{u}\|_\infty \mid x_{\mathbf{u}} \neq y_{\mathbf{u}}\}},.$$

Two configurations are close in this topology if they agree on a large central square. A **subshift** is a closed, shift-invariant subset of some full shift. We call **configurations** of a subshift X the points of X.

Alternatively, subshifts can be defined using forbidden patterns. We call **pattern** any element $P \in \Sigma^U$ where $U \subset \mathbb{Z}^d$ is finite and is the **support** of P, denoted by $\mathrm{supp}(P)$. For a configuration x, we say that P appears in x if there exists $\mathbf{u} \in \mathbb{Z}^d$ such that $\sigma_{\mathbf{u}}(x)_{|U} = P$. Let $\mathcal{F}$ be a collection (finite or not) of patterns. Then the set

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^d} \,\middle|\, \forall P \in \mathcal{F}, P \text{ does not appear in } x \right\}$$

is a subshift. In fact, for any subshift X, there exists a family of patterns $\mathcal{F}$ such that $X = X_{\mathcal{F}}$. A subshift X is a **subshift of finite type** (**SFT**) if there exists a finite $\mathcal{F}$ such that $X = X_{\mathcal{F}}$.

For a given subshift X defined by a fixed family of forbidden patterns $\mathcal{F}$, a pattern $P \in \Sigma^U$ is **locally admissible** if it contains no forbidden patterns $F \in \mathcal{F}$. It is **globally admissible** or **extensible** if it appears in some configuration $x \in X$.

3 Projective Fundamental Group

3.1 Intuitions and definitions

The Projective Fundamental Group, introduced by Geller and Propp [17], resembles the usual fundamental group construction in the topological setting: it is defined through paths, loops, and a homotopy notion. However, instead of directly considering paths between points of the subshift, they are defined between finite patterns with the same support. By doing so, one actually constructs a family of – potentially different – fundamental groups, for each finite support $B \subset \mathbb{Z}^2$. In order to obtain a single group, the projective fundamental group, one takes their inverse (also known as projective) limit. We will construct a subshift by defining a set $\mathcal{T}$ of tiles. A configuration will then be a mapping $x\colon \mathbb{Z}^2 \to \mathcal{T}$ associating a tile to each point of the plane and which verifies some adjacency rules depending on $\mathcal{T}$. Contrary to the usual convention, we will consider that when embedding such a configuration in the Euclidean plane $\mathbb{R}^2$, the tile in position (i,j) is a unit square whose bottom-left corner is placed on (i,j), as opposed to its center. This is merely a discussion about conventions, but it will make some definitions substantially simpler.

Fix a support $B \subset \mathbb{Z}^2$. In what follows B will be called an **aperture window**. Most of the time, we will restrict ourselves to the windows $B_n = [\![-n, n-1]\!]^2$. We choose this asymmetrical window to simplify some definitions, but also for consistency with the aforementioned convention. In any configuration x, the tile $x_{(0,0)}$ in position $(0,0)$ will therefore be seen as the square whose bottom-left (resp. top-right) corner is $(0,0)$ (resp. $(1,1)$).

Consider P, P' two extensible patterns of support B and two points of the grid $\mathbf{v}, \mathbf{v'} \in \mathbb{Z}^2$. A **path** between $(P, \mathbf{v})$ and $(P', \mathbf{v'})$ is a sequence of pairs of patterns and of points of $\mathbb{Z}^2$ (or equivalently, two sequences of the same length). The sequence of points represents an actual, "geometric" path, called its **trajectory**, that is to say a sequence of vertices of $\mathbb{Z}^2$ starting at $\mathbf{v}$ and ending at $\mathbf{v'}$, where consecutive vertices are at euclidean distance exactly 1. The sequence of patterns associates with each one of those vertices $\mathbf{v_t}$ a pattern P_t, that needs to be coherent with the path: when moving to the next vertex $\mathbf{v_{t+1}}$ on the trajectory, the next pattern P_{t+1} needs to be coherent with P_t, that is to say, they should be equal where their supports overlap (see Definition 2 for a precise statement). For example, in the full shift over two symbols $\{0,1\}$, and for $B = B_1$, take the following patterns:

$$P_1 = \left(\boxed{\begin{smallmatrix} 0 & \color{red}{0} \\ 0 & 0 \end{smallmatrix}}, (0,0) \right), \qquad P_2 = \left(\boxed{\begin{smallmatrix} \color{red}{0} & 1 \\ 0 & 1 \end{smallmatrix}}, (1,0) \right), \qquad P_3 = \left(\boxed{\begin{smallmatrix} \color{red}{1} & 1 \\ 1 & 1 \end{smallmatrix}}, (1,0) \right)$$

The tile in position $(0,0)$ is represented in red. The sequence (P_1, P_2) is a valid path, as the overlapping parts of the support are equal in both patterns, but (P_1, P_3) is not because the point $(0,0)$ is tiled by 0 in the first pattern but by 1 in the second one. Moreover, the pattern obtained by "merging" two consecutive patterns also needs to be an extensible pattern.

▶ **Definition 2** (Path). *Let $B \subset \mathbb{Z}^2$ be a finite set, a path of* ***aperture window*** *B is a finite sequence $(P_t, \mathbf{v_t})_{0 \le t \le N}$ such that for any t with $0 \le t \le N$:*

- *P_t is an extensible pattern of X of support $B + \mathbf{v_t}$,*
- *$\mathbf{v_t}$ is adjacent to $\mathbf{v_{t+1}}$, i.e., $\mathbf{d_t} = \mathbf{v_{t+1}} - \mathbf{v_t}$ has euclidean norm exactly 1,*
- *$P_t(u) = P_{t+1}(u)$ for any $u \in B \cap \sigma_{\mathbf{d_t}}(B)$, i.e., consecutive patterns overlap,*
- *the pattern $P_t \cup P_{t+1}$ obtained by merging P_t and P_{t+1} is extensible in X.*

The first and last element of the sequence are respectively called the ***starting point*** *and the* ***ending point*** *of the path. If they are equal, the path is called a* ***loop****. The path $(P_{N-t}, \mathbf{v_{N-t}})_{0 \le t \le N}$ is called its* ***inverse path****. If p is a path, its inverse will be denoted by p^{-1}*

The sequence $(\mathbf{v_t})_{0 \le t \le N}$ is called the ***trajectory*** *of the path.*

Two paths may be composed when the first one ends where the second one starts:

▶ **Definition 3** (Path composition). *Given $p = (P_t, \mathbf{v_t})_{0 \le t \le N}$ and $p' = (P'_t, \mathbf{v'_t})_{0 \le t \le N'}$ two paths such that $(P_N, \mathbf{v_N}) = (P'_0, \mathbf{v'_0})$ we denote by $p * p'$ the path*

$$p * p' = (P_0, \mathbf{v_0}) \dots (P_N, \mathbf{v_N})(P'_1, \mathbf{v'_1}) \dots (P'_{N'}, \mathbf{v'_{N'}}).$$

▶ **Definition 4** (Coherent path). *A path $p = (P_i, \mathbf{v_i})_{i \le N}$ is coherent if all its patterns are equal on the points where their supports overlap, and furthermore, the pattern obtained by merging all the P_i is globally admissible in X. In that case, for any $x \in X$ containing $\bigcup_{i \le N} P_i$, we say that p can be* ***traced*** *in x.*

▶ **Definition 5** (Coherent path decomposition). *A* ***coherent decomposition*** *of a path p is a sequence $p_1, \dots, p_L$ of coherent paths such that $p = p_1 * p_2 \dots * p_L$, and L is called the* ***length*** *of the decomposition.*

One can now define a corresponding homotopy notion: let $p = p_1 * p_2 * p_3$ be a path and suppose that p_2 can be traced in a single configuration $x \in X$. Then, for any p'_2 traced in x with the same starting and ending point as p_2, the path $p_1 * p'_2 * p_3$ is called an **elementary deformation** of p. As paths might consist of a single point, they can be deformed by inserting or removing loops traced in a single configuration at any step.

▶ **Definition 6** (Homotopy). *Two paths p, p' are said to be **homotopic** if there exists a finite sequence of elementary deformations from p to p'. This defines an equivalence relation between paths, and we denote by $[p]$ the equivalence class of p. If p and p' are paths with an aperture window $B \subset \mathbb{Z}^2$, we denote by $p \sim_B p'$ the fact that they are homotopic.*

▶ Remark 7. When two paths are homotopic, they necessarily have the same starting and ending points. When B is clear from the context, we will simply write $p \sim p'$.

With this definition of a path and of homotopy, we can define a fundamental group for each possible aperture window $B \subset \mathbb{Z}^2$.

▶ **Definition 8** (Fundamental Group). *Let X be a SFT, $B \subset \mathbb{Z}^2$ an aperture window, $x_0 \in X$ and $\mathbf{v} \in \mathbb{Z}^2$. The **fundamental group** of X based at $(x_0, \mathbf{v})$ for the aperture window B, denoted by $\pi_1^B(X, (x_0, \mathbf{v}))$, is the group of all the equivalence classes of loops starting and ending at $(x_{0|B}, \mathbf{v})$ for the homotopy equivalence relation, along with the $*$ operation.*

Although our paths follow the $\mathbb{Z}^2$ grid and seem to be discrete and combinatorial objects, it is legitimate to refer to those objects as *homotopy* and *deformations*, which usually suppose some kind of continuity. In fact, this simplification does not entail any loss of generality, compared to paths drawn in $\mathbb{R}^2$, and subshifts seen as $\mathbb{Z}^2$-invariants subsets of $\Sigma^{\mathbb{R}^2}$ (see [17, Subshifts and albums] for more details). In order to obtain a single object associated with the subshift, we get rid of this reference to an aperture window by considering the projective limit of those groups to define the **Projective Fundamental Group** of the subshift.

▶ **Definition 9** (Restriction maps). *For any $B' \subseteq B \subset \mathbb{Z}^2$, the map*

$$\begin{aligned} \mathrm{restr}_{B,B'} : \Sigma^B &\to \Sigma^{B'} \\ P &\mapsto (i \in B' \mapsto P(i)) \end{aligned}$$

*is called the **canonical restriction map** from B to B'. We can naturally extend it to $\bigcup\limits_{\mathbf{v}\in\mathbb{Z}^2} \Sigma^{B+\mathbf{v}}$ so that $\mathrm{supp}(P) = B + \mathbf{v} \implies \mathrm{supp}(\mathrm{restr}_{B,B'}(P)) = B' + \mathbf{v}$.*

Intuitively, these maps simply "forget" some parts of the pattern. We also extend these maps to paths: if $B' \subseteq B$, the image of a path p with aperture window B is a path with the same trajectory with aperture window B', obtained by mapping $\mathrm{restr}_{B,B'}$ element-wise on p.

▶ **Definition 10** (Projective path class). *Let $x, x' \in X$ and $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}^2$. A **projective path class** between $(x, \mathbf{v})$ and $(x', \mathbf{v}')$ is a sequence $([p_n])_{n>0}$ along with the canonical restriction maps, such that p_n is a path of aperture window B_n between $(x_{B_n}, \mathbf{v})$ and $(x'_{B_n}, \mathbf{v}')$, and for each $n > n' > 0$, $\mathrm{restr}_{B_n,B_{n'}}(p_n) \sim_{B_{n'}} p_{n'}$.*

In the case where $(x, \mathbf{v}) = (x', \mathbf{v}')$, we instead say that $([p_n])_{n>0}$ is a projective loop class based at $(x, \mathbf{v})$.

▶ **Definition 11** (Projectively connected subshift). *A subshift X is **projectively connected** if for any two points $x, x' \in X$, there exists a projective path class between $(x, (0, 0))$ and $(x', (0, 0))$.*

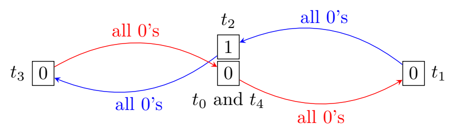
As before, projective loop classes based at the same $(x, \mathbf{v})$ can be concatenated component-wise, to obtain another projective loop class.

▶ **Definition 12** (Projective Fundamental Group). *The **projective fundamental group** based at the point $(x_0, \mathbf{v}) \in X \times \mathbb{Z}^2$ of a subshift X is the group of projective loop classes based at $(x_0, \mathbf{v})$, with the group operation being the component-wise concatenation of projective loop* *classes, and is denoted by* $\pi_1^{proj}(X,(x_0,\mathbf{v}))$*. If* X *is projectively connected, then its projective fundamental group does not depend on the chosen basepoint* $(x_0,\mathbf{v})$*, and we denote it by* $\pi_1^{proj}(X)$*.*

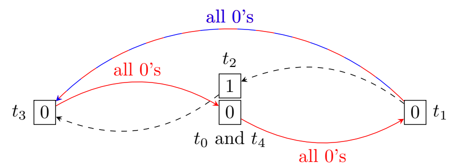
This is a usual construction of what is called a projective (or inverse) limit in category theory. However, we do not use general properties of inverse limits in the rest of the article.

3.2 First example

We slightly modify an example of [17]. Consider the two-dimensional subshift X on the alphabet $\{0,1\}$ of all the configurations containing at most one 1. We show how some paths can be deformed to the trivial path. It is then easy to show that *all* paths are homotopic to the trivial path. Take an aperture window of size 1, *i.e.,* only one cell is visible at a time. Consider the following path p, starting at $(0,(0,0))$ (we see a 0 at the origin of the $\mathbb{Z}^2$ plane). The path then moves in the $\mathbb{Z}^2$ grid while only seeing 0's, and comes back to the origin where it now sees a 1. Then it moves away from the origin while only seeing 0's, and finally comes back to $(0,0)$ with a 0 in the window. For simplicity, we also suppose that the path does not pass through the origin at any other time. To sum up, the path is a loop, starting and ending at $(0,(0,0))$, which only sees 0 along the way except at one time (t_2 on the figure) where it sees a 1 at the origin. This is illustrated in Figure 1a.



(a) Example of a path that cannot be traced in a single configuration.



(b) A homotopic deformation to a path that can entirely be traced in the all-0 configuration.

Figure 1 Example of a path and of a deformation of this path. Notice that the central 0 and 1 windows at t_0 and t_2 are actually located at the same point of the plane, although the figure depicts them on top of each other for the sake of clarity. Red wires can be traced in x_0, and blue wires in x_1. The wire of alternating colours can be traced within both, and so it is both homotopic to the initial path, and to the trivial path.

Let x_0, x_1 respectively be the all-zero configuration, and the configuration containing a 1 at the origin. The path p can be homotopically deformed in the following way: between the times t_1 and t_3, it can be considered to be entirely in x_1. It can thus be deformed in this configuration by completely avoiding the origin, and joining the same points, as in Figure 1b. By definition of x_1, this new path will now see only 0's. The resulting loop then also sees 0's at any point, and so it can be homotopically contracted to the trivial path in the configuration x_0. This proof can be extended to make any 1 on a path "disappear", and so any path can be contracted. In this case, this shows that $\pi_1^{proj}(X,(x_0,(0,0))) = \{e\}$ is trivial, as the same argument works for arbitrary large B_n.

4 Realization of projective fundamental groups

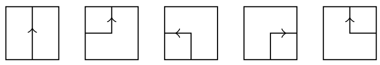
We are now going to prove our main result: any finitely presented group is the fundamental projective group of some SFT.

▶ **Theorem 1.** *Let* $G = \langle S|R\rangle$ *be a finitely presented group. Then, there is a subshift of finite type* X *satisfying:*

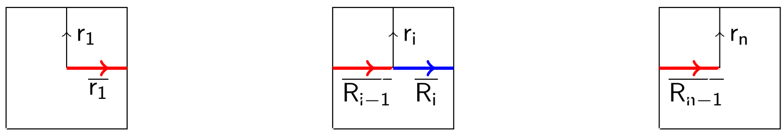
- *X is projectively connected,*
- *the projective fundamental group of X is isomorphic to G.*

4.1 The construction

The subshift X that we construct will informally consist of oriented wires, drawn on an empty background, each wire corresponding to a generator $\mathsf{s} \in S$ of the group $G = \langle S | R \rangle$. We only authorize the wires to go up, perhaps in some kind of "zigzag" manner, but never down or horizontally. More precisely, we define the following tiles: first of all, a tile that we call **empty**, visually represented by $\square$, and we denote by $\mathcal{T}_{\text{empty}}$ the singleton containing this tile. We denote by $x_\square \in X$ the configuration which only contains empty tiles, and its patterns are called **empty patterns**. Then, for each element $\mathsf{s} \in \bar{S} = S \cup \{\mathsf{s}^{-1} | \mathsf{s} \in S\}$, we also consider the set $\mathcal{T}_\mathsf{s}$ of the 5 following tiles:



If $\mathsf{s} \neq \mathsf{s}'$, then $\mathcal{T}_\mathsf{s} \cap \mathcal{T}_{\mathsf{s}'} = \emptyset$. Distinct $\mathcal{T}_\mathsf{s}$ will be represented by wires of different colours in the figures. These tiles will, intuitively, be used to represent generators of the group in valid configurations of X. Finally, we use some other tiles that will play the role of representing the group relations. We can always assume that R contains the trivial relators $\mathsf{s}\mathsf{s}^{-1}$ and $\mathsf{s}^{-1}\mathsf{s}$ for all $\mathsf{s} \in S$. Now, for each relator $\mathsf{r} = \mathsf{r}_1\mathsf{r}_2 \ldots \mathsf{r}_n \in R$, we let $\mathcal{T}_\mathsf{r}$ be the tiles described by Figure 2.



(a) Start **(b)** For $2 \leq i < n$ **(c)** End

Figure 2 The relation tiles.

The wire exiting from the right side of the tile Figure 2a does *not* have the same colour as the one exiting from the top. The former colour is denoted by $\overline{\mathsf{r}_1}$, to differentiate it from the actual r_1 wires. In the other tiles, $\overline{\mathsf{R}_\mathsf{i}} = \overline{\mathsf{r}_1\mathsf{r}_2 \ldots \mathsf{r}_\mathsf{i}}$. Hence, for each relator $\mathsf{r}_1 \ldots \mathsf{r}_\mathsf{n}$, we have one tile of type Figure 2a and one of type Figure 2c, and $n-2$ tiles of type Figure 2b. Tiles belonging to some $\mathcal{T}_r$ are called relation tiles. Note that if $\mathsf{u} \in R$ is such that it is the prefix of two different relators, *i.e.*, there exists $\mathsf{v}, \mathsf{v}' \in \bar{S}^*$ such that $\mathsf{u}\mathsf{v} \in R, \mathsf{u}\mathsf{v}' \in R$ then the colours $\bar{\mathsf{u}}$ are shared by the tiles used to represent those relators and so $\mathcal{T}_{\mathsf{u}\mathsf{v}} \cap \mathcal{T}_{\mathsf{u}\mathsf{v}'} \neq \emptyset$. X is the subshift generated by the tileset $\mathcal{T} = \mathcal{T}_{\text{empty}} \cup \bigcup_{\mathsf{s} \in \bar{S}} \mathcal{T}_\mathsf{s} \cup \bigcup_{\mathsf{r} \in R} \mathcal{T}_\mathsf{r}$ along with the obvious adjacency rules: any wire must be extended, by a wire with the same orientation given by the arrows – *e.g.*, and are forbidden patterns, but is allowed (assuming the two tiles contain a wire of the same colour).

We now formalize what we really mean by a wire.

▶ **Definition 13** (Wire). *A **wire** is a sequence $\mathcal{U} = (T_t, \mathbf{v_t})_{t \in I}, I \subseteq \mathbb{Z}$ a non-necessarily finite interval, of pairs of non-empty tiles and $\mathbb{Z}^2$ points, such that*

- $\|\mathbf{v_{t+1}} - \mathbf{v_t}\|_1 = 1$,
- *The tile T_{t+1} in position $\mathbf{v_{t+1}}$ extends the wire of tile T_t in position $\mathbf{v_t}$: placing a tile above or below another tile does extend it, while placing it on its right or left side does not, although they are valid patterns of X.*

- *$\mathcal{U}$ does not contain two consecutive relation tiles.*

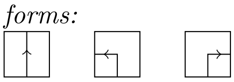
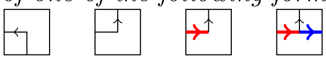
▶ Remark 14. We do not prevent a wire from moving back and forth: it is possible to have $(T_t, \mathbf{v_t}) = (T_{t+2}, \mathbf{v_{t+2}})$.

▶ **Definition 15** (Coherent wire). *We say that a wire is **coherent** if there exists a configuration $x \in X$ such that for any tile $(T_i, \mathbf{v_i})$ of the wire, $x_{\mathbf{v_i}} = T_i$.*

▶ Remark 16. Valid configurations of X can contain non-intersecting infinite wires, and possibly some relation tiles with wires originating from them. Any relation tile belongs to one horizontal line of k relation tiles, corresponding to a valid relator $\mathsf{r_1} \dots \mathsf{r_k}$.

One important concept associated to paths on this subshift is the idea that paths can cross wires. Informally, this is what happens when the window, and in particular, its center, moves from one side to the other of a given wire in a path.

▶ **Definition 17** (Crossing a wire tile). *Let $n > 0$, and let $\mathbf{v}, \mathbf{v'} \in \mathbb{Z}^2$ be two adjacent points, and P, P' two patterns of respective support $\mathbf{v} + B_n, \mathbf{v'} + B_n$ such that $(P, \mathbf{v}), (P', \mathbf{v'})$ is a valid path. For $(i, j) \in B_n$, let $T_{(i,j)}$ be the tile whose bottom-left corner is on (i, j) in P. We say that this path crosses a wire tile if*

- *$\mathbf{v'} - \mathbf{v} = e_0 = (1, 0)$ (resp. $-e_0$) and the tile $T_\mathbf{v}$ (resp. $T_{\mathbf{v}-e_0}$) was of one of the following forms:*
  
- *$\mathbf{v'} - \mathbf{v} = e_1 = (0, 1)$ (resp. $-e_1$) at the next step $t + 1$ and the tile $T_\mathbf{v}$ (resp. $T_{\mathbf{v}-e_1}$) was of one of the following form:*
  

In the following, we let $B_n = \{-n, \dots, n-1\}^2$. Unless stated otherwise, all the aperture windows considered will be of this form.

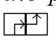
▶ **Definition 18** (Seeing a wire). *A path $p = (P_i, \mathbf{v_i})_{i \le N}$ **sees** a wire $\mathcal{U}$ if there exists a timestep $i \le N$, and $(T_j, \mathbf{v_j}) \in \mathcal{U}$ such that the tile in position $\mathbf{v_j}$ in P_i is T_j.*

▶ **Definition 19** (Crossing a wire). *A path **crosses** a wire if it crosses one of its tiles.*

4.2 Only Crossed Wires Matter

Our final goal is to prove that the projective fundamental group of this subshift X is the group $G = \langle S|R \rangle$. To do so, the idea will be to associate an element of the group to each path, according to the wires that it crosses. The following lemmas can be seen as a procedure to put paths in some kind of normal form via homotopies, depending only the sequence of crossed wires, regardless of the underlying geometry of the path. All the lemmas consider paths that both start and end in empty patterns, but this is not really a restriction as we will later prove that the subshift X is projectively connected, and so we will only consider loops based at $x_\square$. Unless stated otherwise, all the considered paths are using some B_n as aperture window. We start with some easy statements about patterns of support B_n, and the wires they may contain.

▶ **Lemma 20** (Wire Order Lemma). *Let $x \in X$, and let $\mathcal{U}, \mathcal{V}$ be two infinite wires in x. Suppose that $\mathcal{U}, \mathcal{V}$ do not contain relation tiles.*

- *For all $z \in \mathbb{Z}$, there exists between one and two $z^0_\mathcal{U} \in \mathbb{Z}$ such that $\mathcal{U}$ passes through the position $(z^0_\mathcal{U}, z)$. If there are two such $z^0_\mathcal{U}$, then they are necessarily adjacent, e.g.,  side-by-side.*

- *Let $z \in \mathbb{Z}$, and $z^0_{\mathcal{U}}, z^0_{\mathcal{V}} \in \mathbb{Z}$ as in the previous point respectively for $\mathcal{U}$ and $\mathcal{V}$. If $z^0_{\mathcal{U}} < z^0_{\mathcal{V}}$, then for all $z_{\mathcal{U}}, z_{\mathcal{V}}, z \in \mathbb{Z}$ such that $(z_{\mathcal{U}}, z) \in \mathcal{U}, (z_{\mathcal{V}}, z) \in \mathcal{V}$, we have $z_{\mathcal{U}} < z_{\mathcal{V}}$. Intuitively, this means that wires can globally be ordered from left to right.*

If $\mathcal{U}$ or $\mathcal{V}$ contains a relation tile, then the previous claims are true only for z large enough.

▶ Remark 21. Note that the previous lemma is true because we consider wires $\mathcal{U}, \mathcal{V}$ belonging to some configuration. It is clearly false for arbitrary wires.

▶ **Lemma 22.** *Let P be a globally admissible pattern of support B_n for some $n > 0$. Let $\mathcal{U}$ be a wire in P without relation tiles. Suppose that $\mathcal{U}$ passes to the right (resp. left) of $(0, 0)$ in P. Then, $\mathcal{U}$ neither enters nor exits P on its left (resp. right) edge.*

Proof. This directly follows from the fact that no tile contains a horizontal wire, and that B_n is a square. ◀

▶ **Corollary 23.** *If P is a globally admissible pattern that sees a wire $\mathcal{U}$ with no relation tiles, and $x \in X$ is such that $x_{|B_n} = P$, then $\sigma^{4n}_{(0,1)}(x)_{|B_n}$ and $\sigma^{-4n}_{(0,1)}(x)_{|B_n}$ do not see $\mathcal{U}$.*

In order to show that the homotopy class of a path p is indeed only determined by the wires it crosses, we will need several lemmas in which the proof will always be similar: an induction on the length L of a Coherent path decomposition of p:

- for $L = 1$ (*i.e.* p is coherent), we explicitly show how to deform p to obtain the required property.
- for $L = 2$ we use the Path Co-extensibility Lemma to "normalize" both coherent subpaths of p using the base case $L = 1$.
- In general, if $p = p_1 * \ldots * p_N$, we can deform both p_1 and p_2 so that $p \sim p'_1 * p'_2 * \ldots * p_N$, in such a way that we can apply the base case to p'_1, and the induction case to $p'_2 * \ldots * p_N$.

The key step is therefore to properly show how to deal with the case $L = 2$; this is the purpose of the Path Co-extensibility Lemma that we now show, after some preliminary results.

▶ **Lemma 24** (Finite Extension Lemma). *Let P be an extensible finite pattern of X, there exists $x \in X$ containing P, such that x contains a finite number of wires.*

▶ **Definition 25** (Cone). *For $n \in \mathbb{N}$, we define the* ***cones***

$$C^-_n = \{(i, j) \mid j \leq 0, -|j| - n \leq i < |j| + n\} \qquad C^+_n = \{(i, j) \mid j \geq 0, -j - n \leq i < j + n\}$$

We denote $\partial C_n = C_n \cap ((\overline{C_n} + e_0) \cup (\overline{C_n} + e_1))$ the border of a cone.

▶ **Lemma 26** (Extensibility Lemma). *Let $n > 0$. There exists $k > 0$ such that for any $x \in X$, there exists $x' \in X$ with:*

- $x'_{|C^\pm_n} = x_{|C^\pm_n}$
- $x'_{|\sigma_{(0,k)}(C^\mp_n)} = x_{\square|\sigma_{(0,k)}(C^\mp_n)}$

Proof. We prove the case where x' is empty in a cone above the $y = 0$ line, and equal to x below it, the other case being similar. Let r be the length of the longest relator in the finite presentation of $G = \langle S|R\rangle$. Let $W \subset \mathbb{Z}^2$ be the set of positions of tiles that are part of a wire of x that:

- either passes by C^-_n
- or originates from a relation tile which is itself part of a relator intersecting C^-_n.

Now, construct x' as follows:

- for $(i,j) \in C_{n+r}^- \cap W$, set $x'_{(i,j)} = x_{(i,j)}$. The other tiles of C_{n+r}^- are empty.
- for $(i,j) \in \partial C_{n+r}^- \cap W$ and $j < 0$, extend the wire above (i,j) using only tiles ⬏ and ⬐ if $i < 0$, or ⬑ and ⬎ if $i > 0$.
- each wire of W passing by $(i,0)$ with $|i| \leq n+r$ is extended by $n - |i| + r$ tiles ⍐, and then by tiles of the form ⬏ and ⬐ if $i < 0$, or ⬑ and ⬎ if $i \geq 0$.
- all the other tiles are empty.

Then, x' is a valid configuration of X and:

- By definition of W, x', x coincide on C_n^-.
- ∂C_{n+r}^- contains no relation tile, by definition of W and r.
- $(0, n+r+1) + C_n^+$ is empty. See for example Figure 3 or Figure 7.

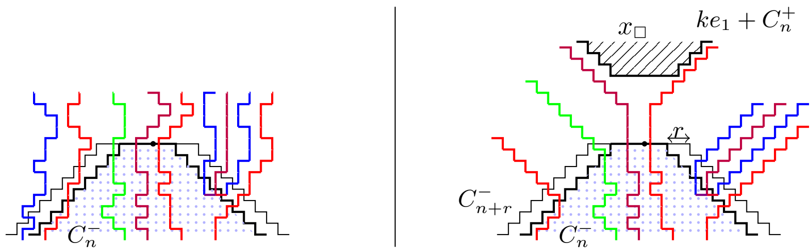


Figure 3 Construction of x' (on the right) from x (on the left). In both figures, the central dot is the origin $(0,0)$.

◀

▶ **Corollary 27** (Path Co-extensibility Lemma). *Let $p = ((P_t, \mathbf{u}_t))_{t \leq N_p}$ and $q = ((Q_t, \mathbf{v}_t))_{t \leq N_q}$ be two paths with the same aperture window B_n, satisfying:*

- *Both p and q are coherent paths*
- $(P_{N_p}, \mathbf{u}_{N_p}) = (Q_0, \mathbf{v}_0)$ *(equivalently, $p * q$ is well-defined)*
- $u_0^1 = v_{N_q}^1$ *(i.e. q ends at the same height as p starts)*

Then, there exists p', q', r paths such that:

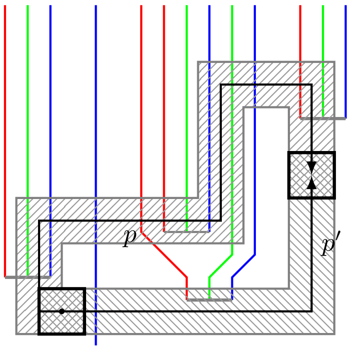
- *r ends on an empty pattern*
- *$p' * r$ and $r^{-1} * q'$ are well-defined and are both coherent paths.*
- *$p \sim p'$ and $q \sim q'$*

Proof. We may assume that $u_0^1 \leq u_{N_p}^1$, *i.e.* the ending point of p is higher than its starting point, the other case being similar. We can also assume that $u_{N_p}^1$ is the highest point in the entire trajectory of both p and q (we can always homotopically deform p and q so that this is true), and up to some shift, we can assume that $\mathbf{u}_{N_p} = (0,0)$. Consider now $P \subset \mathbb{Z}^2$ so that P contains all the P_t and Q_t. Let x_p, x_q be configurations in which p, q can respectively be traced. Take N large enough so that $P \subset C_N^-$. Then, applying the Extensibility Lemma to x_p, N on one hand, x_q, N on the other hand, gives two configurations $x'_p, x'_q \in X$. Let r be the path obtained by moving up for $2N+1$ steps in either x'_p or x'_q, starting from the origin, which is the same path in both cases. Then r satisfies the conditions of Path Co-extensibility Lemma. ◀

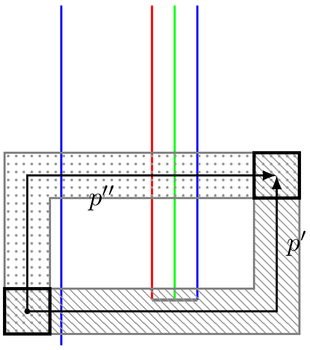
We are now ready to prove the main lemmas needed to show Theorem 1.

▶ **Lemma 28** (No Relation Tile Lemma). *Let p be a path starting and ending on an empty pattern. Then there exists $p' \sim p$ that does not contain any relation tile.*

Proof. As explained above, the proof is by induction on the length of a coherent path decomposition of p. The base case when p is a coherent path is illustrated in Figure 4. See Appendix A for the full proof.



(a) Deformation of p into an L-shaped path p'.



(b) Deformation of p' into p'' to pass above relation tiles.

Figure 4 A coherent path deformed so as not to see relation tiles

◀

▶ **Lemma 29** (Single Wire Lemma)**.** *Let $p = (P_i, \mathbf{v_i})_{0 \leq i \leq N}$ be a path starting and ending with empty patterns. There exists a path p', homotopic to p, such that the union of any two consecutive patterns in p' contains at most a single wire.*

Proof. As for the No Relation Tile Lemma, we illustrate in Figure 5 the case where p is itself coherent. For the full proof, see Appendix A. ◀

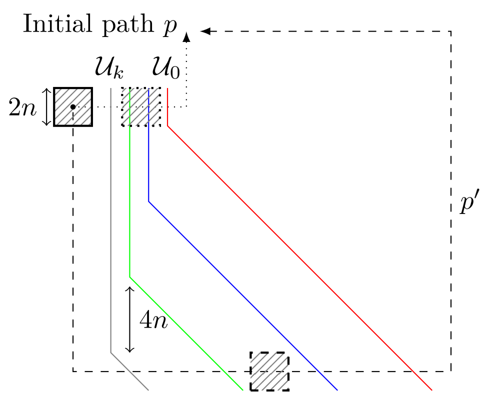


Figure 5 Deformation of p into p' in a single configuration to see only one wire per pattern.

▶ **Lemma 30** (No Uncrossed Wire Lemma)**.** *Let p be a path starting and ending with empty patterns, and $\mathcal{U}$ some wire seen but not crossed by p. There exists a path p', homotopic to p, which does not see $\mathcal{U}$.*

Proof. The idea is that using the previous Single Wire Lemma, we can deal with each wire independently. In particular, the uncrossed wire $\mathcal{U}$ is the only wire seen by some subpath p' of p, and is not seen by p neither before nor after p'. Hence, it suffices to show the result for paths seeing a single wire overall. In that case, one observes that $\mathcal{U}$ has to stay in the same "side" of the aperture window along p', that can therefore be deformed without crossing $\mathcal{U}$ by moving sufficiently far in the opposite direction. For more details, see Appendix A. ◀

▶ **Lemma 31** (Cross Anywhere Lemma). *Let p be a path starting and ending with empty patterns. If p sees no relation tiles, but sees and crosses a single wire $\mathcal{U}$ exactly once, then for all $\mathbf{v} = (v^0, v^1) \in \mathbb{Z}^2$, p is homotopic to a path p' which crosses $\mathcal{U}$ exactly on $\mathbf{v}$.*

Proof. The idea is that if $\mathcal{U}$ exits the aperture window B_n of p in position $(i,j) \in \mathbb{Z}^2$, it can be extended using tiles ⬏ and ⬑, or ⬏ and ⬐, to pass anywhere inside $(i,j) + C_n^-$ or $(i,j) + C_n^+$. The path p can then be deformed to cross it anywhere in those two cones. Using several such deformations, we can deform p so that it crossed $\mathcal{U}$ anywhere in the plane. Note that even if p is initially coherent, it might happen that p' is not, depending on $\mathbf{v}$ and where p initially crossed $\mathcal{U}$. See Appendix A for the complete proof. ◀

4.3 Projective connectedness

▶ **Lemma 32** (Projective connectedness). *X is projectively connected.*

Proof. The proof relies on the Extensibility Lemma. The idea is that starting from any configuration x, there always exists a configuration x' containing a infinite cone (see Definition 25) of $x_\square$, and an infinite cone of x. We can then use this configuration to construct for $n > 0$ a path p_n with aperture window B_n that first moves sufficiently far into the latter cone in x, then to the former cone in the configuration x', and finally comes back to the origin in $x_\square$. See Appendix A for the precise proof. ◀

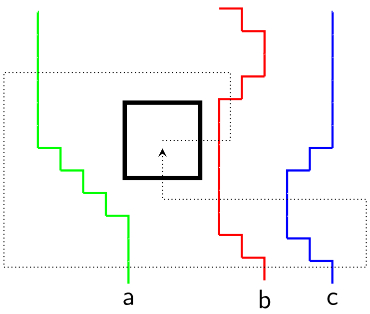
4.4 Computing the projective fundamental group

We can now compute $\pi_1^{proj}(X)$, which is independent of the basepoint since X is projectively connected. Hence, unless stated otherwise, all the loops in this proof are based at $(x_\square, (0,0))$. With any such loop p, we associate a word $[\![p]\!]$ on the alphabet $\bar{S}$ in the following way:

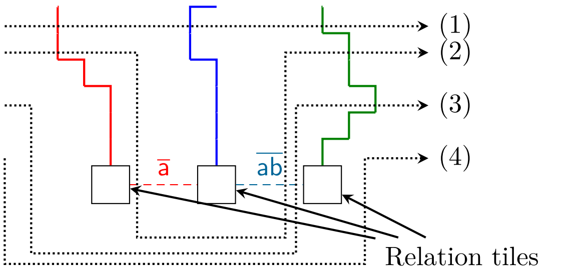
- If p does not cross any wire, we associate the empty word with it, $[\![p]\!] = \varepsilon$.
- If p crosses a single wire $\mathcal{U}$, then:
  - If $\mathcal{U}$ is not a horizontal wire found on a relation tile, and $\mathsf{s} \in \bar{S}$ is the generator corresponding to $\mathcal{U}$ (see Subsection 4.1)
    - ∗ if p crosses it from left to right, or from top to bottom on a tile shaped as ⬏, or from bottom to top on a tile ⬑, then $[\![p]\!] = \mathsf{s} \in \bar{S}$.
    - ∗ if p crosses it in any other direction, we set $[\![p]\!] = \mathsf{s}^{-1} \in \bar{S}$
  - Otherwise, $\mathcal{U}$ is a horizontal wire on a relation tile. Let $\overline{\mathsf{R_i}} = \overline{\mathsf{r_0} \ldots \mathsf{r_i}}$ be its colour.
    - ∗ If it is crossed from top to bottom, then $[\![p]\!] = \mathsf{r_i}^{-1} \ldots \mathsf{r_0}^{-1} \in \bar{S}^*$
    - ∗ Otherwise, $[\![p]\!] = \mathsf{R_i} = \mathsf{r_0} \ldots \mathsf{r_i}$
- If $p = p_1 * p_2$, then $[\![p]\!] = [\![p_1]\!] \cdot [\![p_2]\!] \in \bar{S}^*$ where $\cdot$ represents the concatenation in $\bar{S}^*$.

Some examples are given in Figure 6a and Figure 6b.

For any two words w, w' on $\bar{S}$, we write $\mathsf{w} \equiv \mathsf{w}'$ if they are equal as words on this alphabet, and $\mathsf{w} =_G \mathsf{w}'$ if they represent the same element of the group G. Let $\leftrightarrow_R$ be be the relation defined as the symmetric closure of $\left\{(\mathsf{uwv}, \mathsf{uv}) \mid \mathsf{w} \in R \text{ and } \mathsf{u}, \mathsf{v} \in \left(\bar{S}\right)^*\right\}$, corresponding to the operation of inserting and removing relators to words. We can always suppose that it is reflexive by adding the empty word ε to the relators. We denote $\leftrightarrow_R^*$ its transitive closure.



(a) The word associated with this loop is $\mathsf{bb}^{-1}\mathsf{a}^{-1}\mathsf{abcc}^{-1}\mathsf{b}^{-1} =_G 1_G$.



(b) Widget for the relator $abc = 1_G$. From top to bottom, the words associated with the paths (1) to (4) are respectively $abc = 1_G, aa^{-1}(ab)c = 1_G, (ab)c = 1_G$ and 1_G. For clarity, the relation tiles are not adjacent on the figure

By definition, $\mathsf{w} \leftrightarrow_R^* \mathsf{w}' \iff \mathsf{w} =_G \mathsf{w}'$ (see *e.g.,* [28, Theorem 1.1]). For example, if we take $\mathsf{a} \in S$, we have $\mathsf{aa}^{-1} =_G 1_G$, but $\mathsf{aa}^{-1} \not\equiv \varepsilon$.

In order to prove that the projective fundamental group of this subshift is G, we will prove that the operation $[\![p]\!]$ entirely characterizes a loop up to homotopy, in the sense that loops associated with the same element of G are exactly a projective loop-class:

▶ **Lemma 33** (Homotopic Implies Equal). *For $n > 0$ and any two loops p_n, p'_n starting at $(x_{\square|B_n}, (0,0))$,*

$$p_n \sim_{B_n} p'_n \implies [\![p_n]\!] =_G [\![p'_n]\!]$$

▶ **Lemma 34** (Equal Implies Homotopic). *For any window B_n, and for any pair of loops p_n, p'_n starting at $(x_{\square|B_n}, (0,0))$,*

$$[\![p_n]\!] =_G [\![p'_n]\!] \implies p_n \sim_{B_n} p'_n.$$

The full proofs can be found at Appendix A.

▶ **Theorem 35.** $\pi_1^{proj}(X) = G$

Proof. Let $n > 0$ and let $\Phi_n \colon p \in \pi_1^{B_n}(X, (x_\square, (0,0))) \mapsto [\![p]\!] \in G$ be the function which associates with a loop-class with aperture window B_n the corresponding element of G. The Homotopic Implies Equal and Equal Implies Homotopic show that it is well-defined and injective. Let $[p], [p']$ be two loop-classes based at $(x_{\square|B_n}, (0,0))$. We have shown that $[p] \sim_{B_n} [p'] \iff \Phi_n([p]) =_G \Phi_n([p'])$. Now notice that $\Phi_n([p * p']) =_G \Phi_n(p) \cdot_G \Phi_n(p')$, *i.e.,* Φ_n is a group morphism. To show that it is surjective, let $g \in G$ any element, and $u_1 \dots u_n \in \bar{S}^*$ such that $u_1 \dots u_\ell =_G g$. Let x^g the following configuration:

- For $1 \leq i \leq \ell$ and $j \in \mathbb{Z}$, $x^g(i,j)$ is a tile of type ⇡ and of colour u_i
- Otherwise, $x^g_{(i,j)} = \square$

Now, consider the following loop: define p_n as the loop based at $(x_{\square|B_n}, (0,0))$, which:

- moves left for n steps in $x_\square$
- moves right for $2n + \ell$ steps in x^g – at this point, it sees an empty pattern, after having crossed all the wires of x^g
- comes back to $(0,0)$ in $x_\square$.

By definition, $[\![p_n]\!] \equiv u_1 \dots u_n =_G g$.

Furthermore, notice that for any loop-class $[p_{n+1}]$ based at $(x_{\square|B_{n+1}}, (0,0))$, if p_{n+1} projects down to p then $\Phi_{n+1}([p_{n+1}]) =_G \Phi_n([p])$. This shows that $\pi_1^{proj}(X, (x_\square, (0,0)))$ is isomorphic to G, and the final result follows from the fact that X is projectively connected. ◀

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A Appendix: proofs

The main lemmas of Subsection 4.2 and Subsection 4.3 are proven in this section. As explained in Subsection 4.2, the results are proved by induction on the length of path decompositions.

▶ **Lemma 28** (No Relation Tile Lemma). *Let p be a path starting and ending on an empty pattern. Then there exists $p' \sim p$ that does not contain any relation tile.*

Proof. Let L be the minimal length of a path decomposition of p.

Base case: L = 1 p can be traced entirely in a configuration $x \in X$.

We can assume that x contains a finite number of wires. p being finite, such a configuration exists by the Finite Extension Lemma. Let $(P_N, \mathbf{v_N})$ be the final point of p. Up to a translation of both p and x we can always assume that p starts at $(0,0)$, and without loss of generality, suppose that $\mathbf{v_N}$ is on the right, *i.e.,* it has a non-negative x-coordinate. This is a legitimate assumption, up to considering the path p^{-1} instead of p, which also starts and ends with empty patterns. Deform p into a path p' in x, whose trajectory only consists of moving right, and then up or down, depending on whether $\mathbf{v_N}$ is above or below $(0,0)$. Let $i_{\min}$ (resp. $i_{\max}$) be the leftmost (resp. rightmost) position of a relation tile in x, and let j be the topmost one. We can deform p' as follows:

- Move left until the position $i_{\min} - 2n$ (or don't move if $i_{\min} - n \geq 0$).
- Move up until the position $j + 2n$
- Move right until $i_{\max} + 2n$
- Finally, move to $\mathbf{v_N}$, by moving vertically first and then horizontally.

Let p'' be the resulting path. Then, p'' does not see any relation tile. Figure 4 shows this process in a simple case, with the first and third steps being trivial: Figure 4b, shows how deforming p into p' simplifies the analysis by bounding the positions of the possible relation tiles seen by p', that p'' can then avoid.

Base case: L = 2 $p = p_1 * p_2$

Let $(P_t, \mathbf{v_t})$ be the endpoint of p_1 and the starting point of p_2, with $\mathbf{v_t} = (v_t^0, v_t^1)$. Suppose that $v_t^0 \geq 0, v_t^1 \geq 0$. Let $\mathbf{v}_N$ be the $\mathbb{Z}^2$ point at which p ends – by assumption, the associated pattern P_N is only made of empty tiles. Let $x_1, x_2 \in X$ be two configurations such that p_1, p_2 can respectively be traced entirely within them, and containing a finite number of wires using the Finite Extension Lemma.

In order to be able to use the previous case $L = 1$, we modify the path as follows: consider the path q, traced in x_2, that:

- starts from $(P_t, \mathbf{v_t})$
- follows the inverse trajectory to p_1
- upon reaching $(0,0)$, continues horizontally until it sees an empty pattern (which always eventually happens, as x_2 contains a finite number of wires)

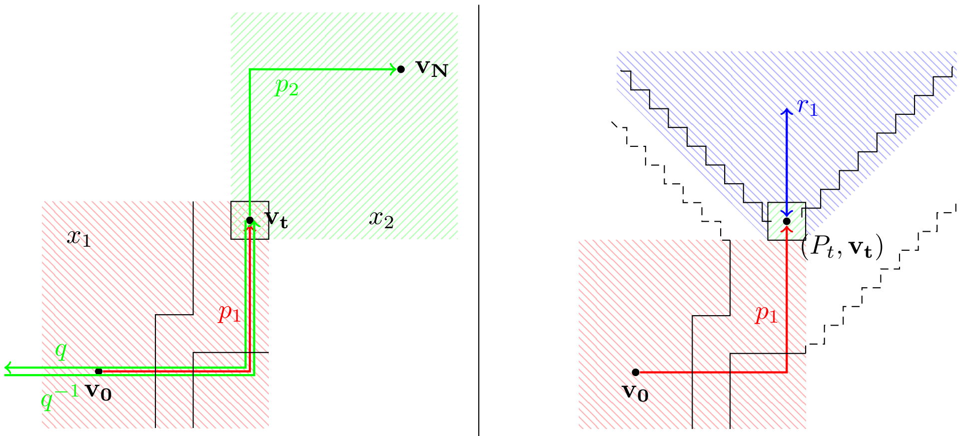
Let $p_1' = p_1 * q$ be and let $p_2' = q^{-1} * p_2$, so that $p = p_1' * p_2'$. By construction, p_2' can be traced entirely within x_2, and so can be appropriately deformed according to the case $L = 1$.

Like p, p_1' has a decomposition of length 2, but we can further simplify it. Indeed, using the Path Co-extensibility Lemma, we obtain a loop $r = r_1 * r_1^{-1}$, based at $(P_t, \mathbf{v_t})$, such that r_1 ends in an empty pattern and each of $p_1 * r_1$ and $r_1^{-1} * q^{-1}$ can be traced within a single configuration. This is enough to prove the case $L = 2$, using three times the case $L = 1$.

The construction is shown in Figure 7.

Finally, we have that

$$p \sim_{B_n} \underbrace{p_1 * r_1}_{\text{traced in } x'_1} \;*\; \underbrace{r_1^{-1} * q}_{\text{traced in } x'_q} \;*\; \underbrace{q^{-1} * p_2}_{\text{traced in } x_2}$$



(a) After inserting q and q^{-1} into $p = p_1 * p_2$ **(b)** The x'_1 configuration

Figure 7 Red paths are traced in x_1, green ones in x_2. Wires are drawn in black.

General case: L > 2 $p = p_1 * \ldots * p_L$.

Consider the timestep t at which p_1 ends and p_2 starts. By definition of a coherent decomposition, there exists $x_2 \in X$ such that p_2 can be entirely traced within x_2. Using the Finite Extension Lemma, we can suppose that x_2 contains finitely many wires. Consider a loop $r = r_1 * r_1^{-1}$ that moves to an empty pattern in x_2 by moving left (this is always possible according to Lemma 22) and then comes back. We have

$$p = p_1 * p_2 \ldots * p_L = \underbrace{p_1 * r_1}_{p'_1} * \underbrace{r_1^{-1} * p_2 \ldots * p_L}_{p'}$$

p'_1 and p' are then respectively paths of length 2 and $L - 1$, and so using the induction hypothesis, they can be deformed so at to avoid any relation tile. ◀

▶ **Lemma 29** (Single Wire Lemma). *Let $p = (P_i, \mathbf{v_i})_{0 \le i \le N}$ be a path starting and ending with empty patterns. There exists a path p', homotopic to p, such that the union of any two consecutive patterns in p' contains at most a single wire.*

Proof. The result is also proved by induction on the length L of a path decomposition of p.

Base case: L = 1 p can be traced entirely in a configuration $x \in X$. Using the No Relation Tile Lemma, we may assume that p does not see any relation tile. Without loss of generality, we may assume that x does not contain any wire that is not seen by p and that p starts at $(0, 0)$ and ends at $\mathbf{v_N} = (v_N^0, v_N^1)$, with $v_N^0 \ge 0, v_N^1 \ge 0$. For simplicity, we assume that the trajectory is made out of two straight segments, so that p first moves horizontally from $(0, 0)$ to $(v_N^0, 0)$ and then vertically to $\mathbf{v_N}$. Let $\mathcal{U}_0, \ldots, \mathcal{U}_k$ be the wires seen from right to left by p (so p sees $\mathcal{U}_k$ first, then $\mathcal{U}_{k-1}$ and so on until $\mathcal{U}_0$).

Now consider a configuration x' satisfying (see Figure 5):

- x' does not contain any other wire than the $\mathcal{U}_i$'s
- for $0 \leq i \leq k$, let $(z_i, -n)$ be the position of the only tile of $\mathcal{U}_i$ whose second coordinate is $-n$, and whose wire enters it from its bottom edge. Then, for $-n - 4ik \leq z \leq -n$, we define $x'(z_i, z)$ to be a tile of the form , and all the tiles of $\mathcal{U}_i$ below that are of the form  and . This uniquely determines all the $\mathcal{U}_i$'s below p.

For $z \in \mathbb{Z}$, no pattern of support B_n centered at $(z, -4n(k+1))$ can see tiles belonging to two different wires at the same time in x'. Therefore, we can deform p in x' into p', where p' starts by moving down for $4n(k+1)$ steps, then right until crossing $\mathcal{U}_0$, and finally up and either right or left as needed to reach $\mathbf{v_N}$. Any two consecutive patterns on this path see at most one wire.

Base case: L = 2 The proof works in exactly the same way as in the proof of the No Relation Tile Lemma.

General case: L > 2 $p = p_1 * \ldots * p_L$.

As before, consider the timestep t at which p_1 ends and p_2 starts. As p_2 is coherent, there exists $x_2 \in X$ such that p_2 can be entirely traced within x_2, and we can assume that x_2 contains finitely many wires. Let r_1 be any path that reaches to an empty pattern in x_2 by moving horizontally left (this always eventually happens, according to Lemma 22). We have

$$p = p_1 * p_2 \ldots * p_L = \underbrace{p_1 * r_1}_{p_1'} * \underbrace{r_1^{-1} * p_2 \ldots * p_L}_{p'}$$

p_1' and p' are respectively paths of length 2 and $L-1$, and the induction hypothesis ensures that they can be homotopically deformed so at not to see $\mathcal{U}$. The resulting path then only sees one wire at a time. ◀

▶ **Lemma 30** (No Uncrossed Wire Lemma). *Let p be a path starting and ending with empty patterns, and $\mathcal{U}$ some wire seen but not crossed by p. There exists a path p', homotopic to p, which does not see $\mathcal{U}$.*

Proof. We proceed by induction on the length L of a coherent decomposition of the path, and we assume that $\mathcal{U}$ is on the right side of the patterns. Using the Single Wire Lemma, we can assume that all the patterns of p contain at most a single wire.

Base case: L = 1 p can be traced entirely in a configuration $x \in X$.

In that case, we can simply deform p in x by changing its trajectory so that it always stays more than n units left from $\mathcal{U}$. This path can then be traced in the configuration x', equal to x except for the tiles of $\mathcal{U}$ in x that are empty tiles in x'.

Base case: L = 2 $p = p_1 * p_2$

Let $(P_t, \mathbf{v_t})$ be the final point of p_1 and the first one of p_2. We also assume that the second coordinate of $\mathbf{v_t} = (v_t^0, v_t^1)$ is non-negative. Let $\mathbf{v_N} = (v_N^0, v_N^1)$ be the final point of the path.

Let $x_1 \in X$ (resp. x_2) be a configuration, containing a minimal number of wires (which exists according to the Finite Extension Lemma), such that p_1 (resp. p_2) can entirely be traced within it. Let $\mathcal{U}$ be the uncrossed wire. We can always assume that $\mathcal{U}$ appears in P_t, otherwise, we could consider p_1 and p_2 separately and apply twice the case $L = 1$.

We deform p_1 into p_1' inside x_1:

- Starting from $(0,0)$, it first moves to the right, until $\mathcal{U}$ appears on the central tile of the pattern seen by p_1.
- It then moves up, left or right, following $\mathcal{U}$: up if the central tile is ⍐, left then up if it is ⬑, and so on.
- Finally, once it attains the height v_t^1, it moves left until $\mathbf{v_t}$ if needed, which takes at most n steps.

We can also deform p_2 into another path p_2' as follows:

- Starting from $\mathbf{v_t}$, move left for $\max(2n, (v_t^0 - v_N^0))$ steps. This ensures that we are far enough so as to not see $\mathcal{U}$ anymore.
- Then, move vertically to height v_N^1.
- Finally, move right until $\mathbf{v_N}$.

Let $\mathbf{w_1}$ be the last point of p_1' before seeing $\mathcal{U}$, and $\mathbf{w_2}$ the first point of p_2' after having seen $\mathcal{U}$ for the last time. The Single Wire Lemma ensures that the patterns seen at both $\mathbf{w}_1$ and $\mathbf{w}_2$ are empty. This gives a decomposition

$$p \sim p_1' * p_2' \sim p_{\text{start}} * p_{\mathcal{U}} * p_{\text{end}}$$

where p_{start} ends at $\mathbf{w_1}$, $p_{\mathcal{U}}$ is the part of the path between $\mathbf{w_1}$ and $\mathbf{w_2}$, and p_{end} starts at $\mathbf{w_2}$.

$p_{\mathcal{U}}$ can be traced entirely in a configuration x_3 whose only wire is $\mathcal{U}$. In this configuration, it can be homotopically deformed to $p'_{\mathcal{U}}$ which never sees $\mathcal{U}$ according to the case $n = 1$.

The final path $p' = p_{\text{start}} * p'_{\mathcal{U}} * p_{\text{end}}$ does not see $\mathcal{U}$.

General case: L > 2 $\quad p = p_1 * \ldots * p_L$

In that case, the proof is exactly the same as in the No Relation Tile Lemma and the Single Wire Lemma: we insert a loop before p_2 starts that extends it, and from a decomposition of length L we obtain two decompositions of length respectively 2 and $L-1$, which are solved inductively. ◀

▶ **Lemma 31** (Cross Anywhere Lemma). *Let p be a path starting and ending with empty patterns. If p sees no relation tiles, but sees and crosses a single wire $\mathcal{U}$ exactly once, then for all $\mathbf{v} = (v^0, v^1) \in \mathbb{Z}^2$, p is homotopic to a path p' which crosses $\mathcal{U}$ exactly on $\mathbf{v}$.*

Proof. Let $p = (P_i, \mathbf{v_i})_{0 \le i \le N}$ be such a path, and let t be the timestep at which p crosses $\mathcal{U}$. Without loss of generality, we can then assume that the wire is crossed from left to right, *i.e.* $\mathcal{U}$ is on the right side of P_{t-1} and on the left side of P_t.

Let x be any configuration containing $P_{t-1} \cup P_t$. We can suppose that $\mathbf{v_t} = e_0 + \mathbf{v_{t-1}}$, by deforming p in x if needed, and that $\mathbf{v_{t-1}} = (0,0)$. Let r_1 be the path starting from $(P_{t-1}, (0,0))$ which moves left for $4n + 2|v^0|$ steps in x, and let $r = r_1 * r_1^{-1}$. Let q_1 be the path starting from $(P_t, (1,0))$ which moves right for $4n + 2|v^0|$ steps in x, and let $q = q_1 * q_1^{-1}$.

We can deform p in x by inserting the loops r and q respectively at the timesteps $t-1$ and t. Using the the No Uncrossed Wire Lemma twice, this path can itself be deformed into $p_{start} * p' * p_{end}$ with $p' = r_1^{-1} * (P_t, (0,0)) * q_1$, and p_{start}, p_{end} paths that only see empty patterns. The trajectory of p' is a straight horizontal line on the $x-$axis of length $8n + 2|v^0| + 1$.

Let x' be the configuration obtained by extending $\mathcal{U}$ as seen by p' using only tiles of the form ⍐. Without loss of generality, suppose that $v^1 \le 0$. We can deform p' in x' so that it moves up for $8n + 2|v^0|$ steps, then right for $8n + 2|v^0| + 1$ as before and finally down to the endpoint of p'. Call p'' the horizontal part of this path. There exists a configuration x'' in which $\mathcal{U}$ passes by $\mathbf{v}$ and in which p' can be traced. Then, p'' can be deformed in x'' to cross $\mathcal{U}$ on $\mathbf{v}$. This finally gives the result. ◀

▶ **Lemma 32** (Projective connectedness). *X is projectively connected.*

Proof. To prove this, it suffices to show that for any configuration $x \in X$, we can find a sequence $(p_n)_{n>0}$ where each $p_n = (P_i^n, \mathbf{v_i}^n)_{0 \le i \le N_n}$ is a path in X_{B_n} between $(x_{|B_n}, (0,0))$ and $(x_{\square|B_n}, (0,0))$, and such that the canonical restriction of each p_{n+1} is homotopic to p_n.

For $n > 0$, let $P_n = x_{|[-n,n-1]\times[-3n,-n-1]}$. Note that all the P_n are included in the cone C_n^-.

Using the Extensibility Lemma with x and C_1^-, we obtain a configuration x' and a bound k and we can then define p_n as follows:

- Move down for $2n$ steps in x. This means that the pattern seen at the end of this part is exactly P_n.
- Move up for $4n + k$ steps in x' so as to reach an empty pattern.
- Come back in $x_\square$.

Now, we need to show that $\rho_n = \mathrm{restr}_{B_{n+1},B_n}(p_{n+1})$ is homotopic to p_n defined in that way, but it is clear as x' does not depend on n. Indeed, ρ_n can be obtained exactly from p_n by:

- adding a trivial loop of length 4, drawn inside x, below the first part of p_n, as p_{n+1} moves down for two more steps than p_n.
- adding a trivial loop of length 8, drawn inside $x_\square$, above the second part of p_n, as p_{n+1} moves up inside x' for 4 more steps than p_n (from $(0,-2n)$ to $(0, 2n+k)$).

Therefore, $\rho_n \sim_{B_n} p_n$, and so every point $x \in X$ is projectively connected to $x_\square$, so X is projectively connected. ◀

▶ **Lemma 33** (Homotopic Implies Equal). *For $n > 0$ and any two loops p_n, p_n' starting at $\big(x_{\square|B_n}, (0,0)\big)$,*

$$p_n \sim_{B_n} p_n' \implies [\![p_n]\!] =_G [\![p_n']\!]$$

Proof. As any two homotopic loops can be obtained from one another by a sequence of elementary deformations, we can restrict ourselves to the special case of a single deformation that is a loop based at $(P_t, \mathbf{v_t})$. By definition, this deformation is made in a single configuration $x \in X$. We consider two disjoint cases, according to the presence of relation tiles in x.

- Suppose that x does not contain any relation tile. Any bi-infinite wire splits the space in two disjoint regions (a “left” one and a “right” one). Each time a loop crosses such a wire, it has to cross it in the other direction to come back to its initial region. Because wires do not intersect, the associated word will be some kind of Dyck word, where each $\mathsf{s} \in \bar{S}$ can act as an opening or a closing bracket (in which case, the associated closing (resp. opening) bracket is s^{-1}), so it is clearly equal to 1_G in G. This is the simple case depicted in Figure 6a.
- Now, suppose that x does contain some relation tiles. In this case, notice that any two relation tiles are either part of the same relator and are therefore linked by a finite sequence of horizontal relation tiles, or they are independent (not linked by any wire). Hence, we can consider each one of those patterns separately. Consider such a pattern, with relation tiles that implement a relator $\mathsf{r} = \mathsf{r_0} \dots \mathsf{r_k} \in R$, and a configuration x' that only contains this pattern. Figure 6b represents this in a configuration corresponding to relation $\mathsf{abc} = 1$.

We show that, due to how $[\![\cdot]\!]$ has been defined, all the homotopy-equivalent paths in x' are associated with the same element of G. Let $\mathcal{U}_0, \dots, \mathcal{U}_k$ be the wires corresponding respectively to $\mathsf{r_0}, \dots, \mathsf{r_k}$, and suppose that the relation tiles in x' are placed on $(0,0), \dots, (k,0)$. We will show that for any p joining $(0,0)$ to $(k+1,0)$ in x', $[\![p]\!] =_G 1_G$. Let $\mathcal{R} \subset \mathbb{Z}^2$ be the set of points above the $(\mathbb{Z},1)$ line and between $\mathcal{U}_0$ and $\mathcal{U}_k$. We can always suppose that no wire is crossed consecutively in opposite directions, as the word associated to a path that crosses a wire in a direction and immediately crosses it in the other direction is $\mathsf{ss}^{-1} =_G 1_G$ for some $\mathsf{s} \in \bar{S}^*$. We can also suppose that p only enters and then leaves $\mathcal{R}$ once. Otherwise, we can simply split it into several such paths and prove the claim for each of them independently.

- If p crosses $\mathcal{U}_0, \dots, \mathcal{U}_k$, then $[\![p]\!] \equiv \mathsf{r_0} \dots \mathsf{r_k} =_G 1_G$ by definition.
- If p crosses $\mathcal{U}_0, \dots, \mathcal{U}_i, \mathcal{U}_{\overline{\mathsf{r_0 \dots r_i}}}$, where $\mathcal{U}_{\overline{\mathsf{r_0 \dots r_i}}}$ is a wire of a relation tile which is necessarily crossed from top to bottom, by definition, $[\![p]\!] \equiv \mathsf{r_0} \dots \mathsf{r_i}(\mathsf{r_i}^{-1} \dots \mathsf{r_0}^{-1}) =_G 1_G$
- Otherwise, p crosses $\mathcal{U}_{\overline{\mathsf{r_0 \dots r_i}}}, \mathcal{U}_{i+1}, \dots, \mathcal{U}_j, \mathcal{U}_{\overline{\mathsf{r_0 \dots r_j}}}$, the first relation tile being crossed from bottom to top to enter $\mathcal{R}$ and the last one being crossed from top to bottom to exit it. By definition, $[\![p]\!] \equiv (\mathsf{r_0} \dots \mathsf{r_i})\mathsf{r_{i+1}} \dots \mathsf{r_j}(\mathsf{r_j}^{-1} \dots \mathsf{r_0}^{-1}) =_G 1_G$

This shows that all the paths traced in a single configuration are associated with the same element of the group G. As all homotopies are deformations in a given configuration, this implies that for any homotopically equivalent paths p, p', we have $[\![p]\!] =_G [\![p']\!]$. ◀

▶ **Lemma 34** (Equal Implies Homotopic)**.** *For any window B_n, and for any pair of loops p_n, p'_n starting at $(x_{\square|B_n}, (0,0))$,*

$$[\![p_n]\!] =_G [\![p'_n]\!] \implies p_n \sim_{B_n} p'_n.$$

Proof. Using the No Relation Tile Lemma, we can always start by deforming p_n and p'_n so that they do not see any relation tile. As each elementary deformation is by definition occurring in some given configuration, Homotopic Implies Equal ensures that we still have $[\![p_n]\!] =_G [\![p'_n]\!]$. We will first prove that $[\![p_n]\!] \equiv [\![p'_n]\!] \implies p_n \sim_{B_n} p'_n$, which is a stronger assumption. Next, we prove that given p_n and p'_n with $[\![p_n]\!] =_G [\![p'_n]\!]$, there exists a loop p''_n such that $p_n \sim_{B_n} p''_n$ and $[\![p''_n]\!] \equiv [\![p'_n]\!]$. We then have that $p''_n \sim_{B_n} p'_n$ according to the first part of the proof, and so $p_n \sim_{B_n} p'_n$.

- We show that $[\![p_n]\!] \equiv [\![p'_n]\!] \implies p_n \sim_{B_n} p'_n$. The paths p_n and p'_n can be deformed using the No Uncrossed Wire Lemma so that they cross all the wires that they see. The Single Wire Lemma can then be used to deform them so that there is at most one of those wires per pattern. Let $\hat{p_n}$ and $\hat{p'_n}$ be the resulting paths, which by assumption cross the same wires. Using the Cross Anywhere Lemma for each of those crossed wires, we can finally deform $\hat{p_n}$ into $\hat{p'_n}$, and so $p_n \sim_{B_n} p'_n$.
- Now, we show the existence of a loop p''_n satisfying $p_n \sim_{B_n} p''_n$ and $[\![p''_n]\!] \equiv [\![p'_n]\!]$. By definition of $=_G$, there exists a finite sequence $(\mathsf{u_i})_{0 \le i \le N}$ of words on the alphabet $\bar{S}$ such that $[\![p_n]\!] \equiv \mathsf{u_0}, [\![p'_n]\!] \equiv \mathsf{u_N}$, and for all $i < N$, $\mathsf{u_i} \leftrightarrow_R \mathsf{u_{i+1}}$. To prove the result, it is therefore enough to show that for any word v such that $[\![p_n]\!] \leftrightarrow_R \mathsf{v}$, we can deform p_n in another loop p^{v}_n such that $[\![p^{\mathsf{v}}_n]\!] \equiv \mathsf{v}$.
  Suppose that v is obtained from $[\![p_n]\!]$ by deleting a relator. More formally, there exists words $\mathsf{u_1}, \mathsf{u_2}$ and a relator $\mathsf{r} \in R$ such that $\mathsf{v} \equiv \mathsf{u_1 u_2}$ and $[\![p_n]\!] \equiv \mathsf{u_1 r u_2}$. Using the Single Wire Lemma followed by the No Uncrossed Wire Lemma, we obtain a loop $q \sim p_n$, such that q crosses exactly wires of the same type as p_n, but it only ever sees one wire at a time, and crosses all the wires that it sees. The Cross Anywhere Lemma then ensures that we can deform q into a loop that crosses wires corresponding to the letters of $\mathsf{u_1 r u_2}$,

in order, on a horizontal line. Let p_{u_1} (resp. $p_{\mathsf{r}}, p_{\mathsf{u}_2}$) be the part of this path which crosses the wires corresponding to u_1 (resp. r, u_2), starting and ending with empty patterns. Let $x_{\mathsf{r}} \in X$ be such that p_{r} can be traced in x_{r}, and in which all those wires originate from the same set of relation tiles (see Figure 6b). We can then deform p_{r} in x_{r} into a path p'_{r} that passes below the relation tiles. The resulting path $p_n^{\mathsf{v}} = p_{\mathsf{u}_1} * p'_{\mathsf{r}} * p_{\mathsf{u}_2}$ is then a solution.

◀