Computability of extender sets in multidimensional subshifts

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Abstract

Subshifts are colorings of $\mathbb{Z}^d$ defined by families of forbidden patterns. Given a subshift and a finite pattern, its extender set is the set of admissible completions of this pattern. It has been conjectured that the behavior of extender sets, and in particular their growth called *extender entropy* ([9]), could provide a way to separate the classes of sofic and effective subshifts. We prove here that both classes have the same possible extender entropies: exactly the Π_3 real numbers of $[0, +\infty)$. We also consider computational properties of extender entropies for subshifts with some language or dynamical properties: computable language, minimal and some mixing properties.

2012 ACM Subject Classification Mathematics of computing → Discrete mathematics; Theory of computation

Keywords and phrases extender sets, extender entropy, computability, sofic shifts, subshifts, tilings, symbolic dynamics

Funding This research was partially funded by ANR JCJC 2019 19-CE48-0007-01.

1 Introduction

Multidimensional subshifts are sets of colorings of $\mathbb{Z}^d$ where a family of patterns, *i.e.* colorings of finite portions of $\mathbb{Z}^d$, have been forbidden. They have been introduced originally to discretize continuous dynamical systems [19]. One of the main families of subshifts that has been studied is the class of *subshifts of finite type* (SFTs), which can be defined with a finite family of forbidden patterns. This class has independently been introduced under the formalism of Wang tiles [23] in dimension 2 in order to study fragments of second order logic.

In dimension 1 the study of SFTs is done mainly through the study of its defining graph. They share most of their properties with the class of *sofic shifts* [24], which can be obtained as letter-to-letter projections of SFTs. These can also be characterized as the biinfinite walks on some finite automaton. In dimension 2 and higher the main tool in the study of subshifts becomes computability theory. This has led to the introduction of a new class of subshifts, the *effective subshifts*, which can be defined by computably enumerable families of forbidden patterns [12]. Many examples of effective subshifts with interesting properties have been proven to be sofic, such as substitutive subshifts [20], or even effective subshifts on $\{\mathtt{0}, \mathtt{1}\}$ whose densities of symbols $\mathtt{1}$ are sublinear [5]. It turns out that sofic subshifts of dimension $d+1$ capture all the behaviors of effective shifts of dimension d [12, 7, 2], which makes it hard to distinguish the two classes in dimension $d \geq 2$.

An important question in symbolic dynamics it thus to find criteria separating the two classes [14]. All the arguments used to prove some cases of non-soficity that are known by the authors all revolve around a counting argument: only a linear amount of information may cross the border of an $n \times n$ square pattern. The most recent argument in this vein uses resource-bounded Kolmogorov complexity [6]. These arguments all depend on the structure of $\mathbb{Z}^d$ and fail on groups which are not amenable, as in such cases the border of a pattern may carry as much information as its interior. There even exist groups where the class of sofic shifts and effective shifts coincide [3].

One way to extend these counting arguments is through the notion of *extender sets* of patterns [14]: the extender set of a pattern p is the set of all configurations with a p-shaped hole that may extend p. In dimension 1, being sofic means that there is a constant bounding the number of extender sets for any pattern shape. In higher dimensions, this no longer holds, but one can look at the growth of the number of extender sets. For SFTs, a large enough boundary (depending only on the size of the forbidden patterns) entirely determines the extender set of a given pattern, which implies that the number of extender sets of an SFT cannot grow too quickly. In [21] it was proved that a subshift whose number of extender sets for patterns of size n^d is bounded by n must be sofic.

The study of the growth rate of the number of extender sets can be done asymptotically through the notion of the *extender entropy*, which is defined in a similar way to the classical notion of topological entropy [17]. Extender entropies in fact relate to the to notion of follower entropies [4], but are more robust in the sense that the extender entropy of a subshift is a conjugacy invariant.

In this paper, we aim at a better understanding of this quantity, and we achieve characterizations of the possible extender entropies in terms of computability, in the same vein as recent results on conjugacy invariants [13, 18].

▶ **Theorem A.** *The set of extender entropies of $\mathbb{Z}$ effective subshifts is exactly $\Pi_3 \cap [0, +\infty)$.*

▶ **Theorem B.** *The set of extender entropies of $\mathbb{Z}^2$ sofic subshifts is exactly $\Pi_3 \cap [0, +\infty)$.*

This result also disproves a conjecture made in [14] stating that being sofic implies having extender entropy zero: Π_3 numbers are dense in $[0, +\infty)$. It also shows that the value of extender entropy does not allow one to separate sofic and effective shifts, since both have the same possible values. Even in the case of subshifts with computable languages, extender entropies span a large number of possible values:

▶ **Theorem C.** *The set of extender entropies of $\mathbb{Z}^2$ (sofic) subshifts with computable language is exactly $\Pi_2 \cap [0, +\infty)$.*

Finally, we also study extender entropies of subshifts constrained by some dynamical assumptions, such as minimality or mixingness. What is known by the authors at this stage can be summed up by the following table:

	$\mathbb{Z}$	$\mathbb{Z}^d, d \geq 2$
SFT	$\{0\}$ (Folklore: see Proposition 6)	
Sofic	$\{0\}$ ([8, Theorem 1.1])	Π_3 (Theorem B)
Effective	Π_3 (Theorem A)	
Computable	Π_2 (Theorem C)	
Effective and minimal	Π_1 (Corollary 13)	
Effective and 1-Mixing/Block-Gluing	Π_3 (Proposition 15)	Π_3 (Proposition 16)

Results in the paper are proved for $d = 2$, and generalizations to $d > 2$ follow from Claim 7, as the free lift of a sofic (resp. effective, block-gluing, computable) subshift is still sofic (resp. effective, block-gluing, computable).

2 Definitions

2.1 Subshifts

Let $\mathcal{A}$ denote a finite set of symbols and $d \in \mathbb{N}$ the dimension. A **configuration** is a coloring $x \in \mathcal{A}^{\mathbb{Z}^d}$, and the color of x at position $p \in \mathbb{Z}^d$ is denoted by x_p. A (d-dimensional) **pattern** over $\mathcal{A}$ is a coloring $w \in \mathcal{A}^P$ for some set $P \subseteq \mathbb{Z}^d$ called its **support**[1]. For any pattern w over $\mathcal{A}$ of support P, we say that w **appears** in a configuration x (and we denote $w \sqsubseteq x$) if there exists $p_0 \in \mathbb{Z}^d$ such that $w_p = x_{p+p_0}$ for all $p \in P$.

We sometimes consider patterns or configuration by their restriction: for $S \subseteq \mathbb{Z}^d$ either finite or infinite, and $x \in \mathcal{A}^{\mathbb{Z}^d}$ a configuration (resp. w a pattern), we denote by $x|_S$ (resp. $w|_S$) the coloring of $\mathcal{A}^S$ it induces on S.

▶ **Definition 1** (Subshift). *For any family of finite patterns $\mathcal{F}$, we define*

$$X_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} \mid \forall w \in \mathcal{F},\, w \not\sqsubseteq x \right\}$$

*A set $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ is called a **subshift** if it is equal to some $X_{\mathcal{F}}$.*

Subshifts can also be defined via their topological properties. Let us endow $\mathcal{A}^{\mathbb{Z}^d}$ with the product discrete topology. Given the **shift functions** $(\sigma^t)_{t\in\mathbb{Z}^d}$ defined as $(\sigma^t(x))_p = x_{p+t}$, subshifts are the closed and shift-invariant (*i.e.* for every $t \in \mathbb{Z}^d$, $\sigma^t(X) = X$) subsets of $\mathcal{A}^{\mathbb{Z}^d}$.

Given a subshift X and a finite support $P \subseteq \mathbb{Z}^d$, we define $\mathcal{L}_P(X)$ as the set of patterns w of support P that appear in some configuration $x \in X$. We define the **language** of X as $\mathcal{L}(X) = \bigcup_{P \subseteq \mathbb{Z}^d \text{ finite}} \mathcal{L}_P(X)$. For a pattern w, we also say that it is **globally admissible** in X if it appears in some configuration of X. Slightly abusing notations, we denote $\mathcal{L}_n(X) = \mathcal{L}_{[\![0,n-1]\!]^d}(X)$ for $n \in \mathbb{N}$.

We say that a subshift is a **subshift of finite type** (SFT) if it is equal to some $X_{\mathcal{F}}$ for some finite family $\mathcal{F}$ of forbidden finite patterns. We say that a subshift X is **effective** if it is equal to some $X_{\mathcal{F}}$ for $\mathcal{F}$ a computably enumerable family of finite forbidden patterns $\mathcal{F}$. We say that a subshift X is **computable** if $\mathcal{L}(X)$ is computable. We say that a subshift Y is **sofic** if it is equal to $\varphi(X)$ for X an SFT and $\varphi : X \to Y$ a **factor map** (*i.e.* a continuous and shift-commuting function).

SFTs are of course sofic subshifts, sofic subshifts are effective, and computable subshifts are effective too. On the other directions, let us define the two following **lifts**: given a $\mathbb{Z}$ subshift $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$, consider

- *the periodic lift* $X^{\uparrow} = \{y \in \mathcal{A}^{\mathbb{Z}^2} \mid \exists x \in X, \forall i \in \mathbb{Z},\, y_{\mathbb{Z}\times\{i\}} = x\}$;
- *the free lift* $X^{\Uparrow} = \{y \in \mathcal{A}^{\mathbb{Z}^2} \mid \forall i \in \mathbb{Z},\, y_{\mathbb{Z}\times\{i\}} \in X\}$.

(The periodic lift is easily defined on configurations) Then the following result holds:

▶ **Theorem 2** ([12], [2, Theorem 3.1], [7, Theorem 10]). *If X is an effective $\mathbb{Z}$ subshift, then $X^{\uparrow}$ is a sofic $\mathbb{Z}^2$ subshift.*

To help with later proofs, we introduce the following notation: for $I, J \subseteq \mathbb{Z}$ two non-necessarily finite intervals, and $w \in \mathcal{L}_{I\times J}(X^{\uparrow})$, we denote any row of w by $w^{\downarrow} = w_{|I\times\{j\}}$ for

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[1] It will sometimes be convenient to consider patterns only up to translation of their support. Usually, context will make it clear whether we consider two patterns equal up to a $\mathbb{Z}^d$ translation as being the same or not.

any $j \in J$. As $w \sqsubseteq X^{\uparrow}$, this does not depend on the chosen $j \in J$ and so $w^{\downarrow}$ is well-defined. For any $z \in \mathcal{A}^{\mathbb{Z}}$, we have $(z^{\uparrow})^{\downarrow} = z$. Finally, for $(L_i)_{i\in I}$ a family of sets and $J \subseteq I$, we denote by $\pi_{L_{j_1}\times L_{j_2}\times\ldots} : \prod_{i\in I} L_i \mapsto \prod_{j\in J} L_j$ the cartesian projection, which will in particular be used on subshifts on different layers.

2.2 Extender sets and entropy

▶ **Definition 3** (Extender set). *For $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ a d-dimensional subshift, $P \subseteq \mathbb{Z}^d$ and $w \in \mathcal{A}^P$ a pattern of support P, the **extender set** of w is the set*

$$E_X(w) = \{x \in \mathcal{A}^{\mathbb{Z}^d\setminus P} \mid x \sqcup w \in X\},$$

where $(x \sqcup w)_p = w_p$ if $p \in P$ and $(x \sqcup w)_p = x_p$ otherwise.

In other words, $E_X(w)$ is the set of all possible valid "completions" of the pattern w in X. For example, for two patterns with the same support w, w', we have $E_X(w) \subseteq E_X(w')$ if and only if the pattern w can be replaced by w' every time it appears in any configuration of X.

In the case of $\mathbb{Z}$ subshifts, extender sets are similar to the more classical notions of *follower* (resp. *predecessor*) *sets*, which are the set of right-infinite (resp. left-infinite) words that complete a finite given pattern. We introduce these notions more formally in Section 8.

The traditional notion of complexity in subshifts is called **pattern complexity**, and is defined by $N_X(n) = \mathcal{L}_n(X)$. The asymptotic growth of $|N_X(n)|$ is called the **topological entropy**, and is properly defined because $|N_X(\cdot)|$ is a submultiplicative function.

Adopting a similar approach, for X a $\mathbb{Z}^d$ subshift, we denote $E_X(n) = \{E_X(w) \mid w \in \mathcal{L}_n(X)\}$ its set of extender sets. Using this, [8] defines the **extender set sequence** of X as $(|E_X(n)|)_{n\in\mathbb{N}}$. From this sequence, one can derive[2] a notion of entropy :

▶ **Definition 4** (Extender entropy, [9, Definition 2.17]). *For a $\mathbb{Z}^d$ subshift X, its **extender entropy** is*

$$h_E(X) = \lim_{n\to+\infty} \frac{\log |E_X(n)|}{n^d}$$

The limit actually exists because the map $|E_X(\cdot)|$ is sub-multiplicative: more generally, it verifies the conditions of the Ornstein-Weiss lemma from [15].

Examples

1. Let us consider $X = \mathcal{A}^{\mathbb{Z}^d}$ some full-shift in dimension d. Then X has maximal topological entropy, but $h_E(X) = 0$: indeed, for any two patterns $w, w' \in \mathcal{L}_n(X)$, we have $E_X(w) = E_X(w') = \{\mathcal{A}^{\mathbb{Z}^d\setminus[\![0,n-1]\!]^d}\}$; which implies that $|E_X(n)| = 1$ for every $n \in \mathbb{N}$.

   In particular, this shows that contrary to the usual topological entropy, extender entropy is not increasing with (subshift) inclusion.
2. On the other hand, let us consider X a (strongly) periodic subshift. In other words, we assume there exists $p_1, \ldots, p_d \in \mathbb{N}$ such that, for every $x \in X$ and $i \leq d$, we have $\sigma^{p_i\cdot e_i}(x) = x$. Then X has zero topological entropy, and we also have $h_E(X) = 0$. Indeed, for $n \geq \max p_i$ and $w \in \mathcal{L}_n(X)$, we have $E_X(w) \simeq \{w\}$. Then, for any $w' \neq w$, we have $E_X(w) \neq E_X(w')$, and so $|E_X(n)| \leq |\mathcal{L}_n(X)| \leq p\mathcal{A}^p$, so $|E_X(\cdot)|$ is eventually constant.

[2] The authors define it for the case of $\mathbb{Z}$ subshifts only, but the definition makes sense for higher dimensional shifts.

Some properties

The following results were proved in [9] in the context of $\mathbb{Z}$ subshifts, but easily generalize to $\mathbb{Z}^d$ subshifts:

▶ **Theorem 5.** *The following facts hold for one-dimensional subshifts:*
- h_E *is a conjugacy invariant.*
- h_E *is not necessarily decreasing under factor map.*
- h_E *is additive under product (i.e. for* X, Y *two subshifts,* $h_E(X \times Y) = h_E(X) + h_E(Y)$*).*

This paper studies the potential values of $h_E(X)$ for various classes of multidimensional subshifts, depending on their computational/topological properties. We can already formulate the following property, which is folklore:

▶ **Proposition 6** ([14, Section 2])**.** *Let* X *a* d*-dimensional SFT. Then* $h_E(X) = 0$.

Sketch of proof. In an SFT of neighborhood 1, the extender set of a pattern $w \in \mathcal{A}^{[\![0,n-1]\!]^d}$ is entirely determined by its border, and there are at most $2^{O(n^{d-1})}$ such borders. ◀

2.3 Computability notions

2.3.1 Arithmetical hierarchy

The **arithmetical hierarchy** [22, Chapter 4] stratifies formulas of first-order logic by the number of their alternating unbounded quantifiers: for $n \in \mathbb{N}$, define

$$\Pi_n^0 = \{\forall k_1, \exists k_2, \forall k_3, \dots \ \phi(k_1, \dots, k_n) \mid \phi \text{ only contains bounded quantifiers } \}$$
$$\Sigma_n^0 = \{\exists k_1, \forall k_2, \exists k_3, \dots \ \phi(k_1, \dots, k_n) \mid \phi \text{ only contains bounded quantifiers } \}$$

We say that a decision problem is in Π_n^0 (resp. Σ_n^0) if its set of solutions is described by a Π_n^0 (resp. Σ_n^0) formula: in other words, $\Pi_0^0 = \Sigma_0^0$ corresponds to the set of computable decision problems; Σ_0^1 is the set of computably enumerable decision problems, etc...

2.3.2 Arithmetical hierarchy of real numbers

The **arithmetical hierarchy of real numbers** [25] stratifies real numbers depending on the number of alternating limit operations required to define them. In what follows, we say that a sequence of rationals $(\beta_i)_{i\in I} \in \mathbb{Q}^I$ (for I any product $I_1 \times \dots \times I_n$ of intervals of integers) is computable if the function $i \mapsto \beta_i$ is a total computable function.

We denote $\Pi_0 = \Sigma_0$ the set of computable real numbers, *i.e.* the set of real numbers α for which there exists some computable sequence $(\beta_i)_{i\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ such that $\alpha = \lim \beta_i$ and $\forall i \in \mathbb{N}, |\alpha - \beta_i| \leq 2^{-i}$. In other words, α is computable if one can computably approximate it up to arbitrary precision.

We then define for $n \geq 1$:

$$\Pi_n = \left\{ \inf_{k_1\in\mathbb{N}} \sup_{k_2\in\mathbb{N}} \inf_{k_3\in\mathbb{N}} \dots \ \beta_{k_1,\dots,k_n} \mid (\beta_{k_1,\dots,k_n})_{k_1,\dots,k_n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}^n} \text{ is computable} \right\}$$
$$\Sigma_n = \left\{ \sup_{k_1\in\mathbb{N}} \inf_{k_2\in\mathbb{N}} \sup_{k_3\in\mathbb{N}} \dots \ \beta_{k_1,\dots,k_n} \mid (\beta_{k_1,\dots,k_n})_{k_1,\dots,k_n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}^n} \text{ is computable} \right\}$$

We can show (see *e.g.* [25]) that $(\Pi_n \cup \Sigma_n) \subsetneq \Pi_{n+1} \cap \Sigma_{n+1}$.

3 Elementary constructions on extender sets

The free lift

We use this construction to generalize results on $\mathbb{Z}$ or $\mathbb{Z}^2$ subshifts to higher dimensions:

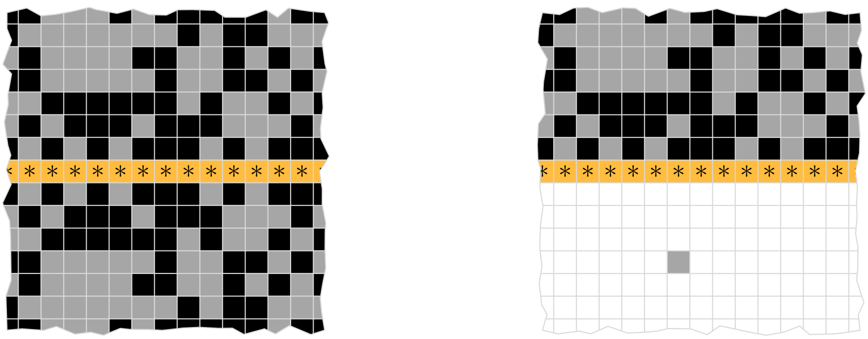
▷ Claim 7. For a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ any subshift, $h_E(X) = h_E(X^{\Uparrow})$.

Proof. Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$. Since the d-dimensional hyperplane of $\mathbb{Z}^{d+1}$ contain independent configurations in $X^{\Uparrow} \subseteq \mathcal{A}^{\mathbb{Z}^{d+1}}$, we have $|E_{X^{\Uparrow}}(n)| = |E_X(n)|^n$, so that $h_E(X^{\Uparrow}) = h_E(X)$. ◀

The (semi)-mirror construction

▷ Claim. Let Y be any $\mathbb{Z}^2$ sofic subshift over an alphabet $\mathcal{A}$. There exists a $\mathbb{Z}^2$ sofic subshift Y_{mirror} such that $h_E(Y_{\text{mirror}}) = h(Y)$.

A natural idea is the *mirror construction*: a line of some special symbol $*$ separates two half-planes; the upper half-plane contains a half-configuration of Y, while the lower half-plane contains its reflection of by the line of $*$. This mirror construction generates a subshift Y' such that $h_E(Y') = h(Y)$, but that unfortunately is not always sofic.



(a) The (classical) mirror shift **(b)** The semi-mirror shift

Figure 1 Example configurations of the mirror and semi-mirror subshifts. The blank symbols $\sqcup$ of the semi-mirror are represented as white □.

To solve this, the *the semi-mirror with large discrepancy* from [6, Example 5″] reflects a single symbol instead of the whole upper-plane:

Sketch of proof. For $\mathcal{A}' = \mathcal{A} \cup \{\sqcup, *\}$, define Y_{mirror} over the alphabet $\mathcal{A}'$ as follows:

- Symbols $*$ must be aligned in a row, and there is at most one such row per configuration.
- If a row of $*$ appears in a configuration x, then the lower half-place is entirely $\sqcup$, with the exception of at most a single position colored by a symbol of $\mathcal{A}$; and the upper half-plane must appear in a configuration of Y.
- If $x_{i,j} = *$ and $x_{i,j-k} \in \mathcal{A}$ for some $i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{N}$, then $x_{i,j+k} = x_{i,j-k}$. In other words, the only symbol of $\mathcal{A}$ in the lower half-plane must be the mirror of the same symbol in the upper half-plane, as reflected by the horizontal row of $*$ symbols.

Then we claim that $h_E(Y_{\text{mirror}}) = h(Y)$. Indeed, any two distinct patterns of Y must appear in Y_{mirror} and have distinct extender sets. ◀

This construction shows that there exists subshifts whose extender entropy is arbitrary; and since by [13] every Π_1 real number is the topological entropy of some (SFT, thus) sofic subshift, every Π_1 real number can be realized as the extender entropy of some sofic subshift. In particular, this already disproves the conjecture from [14] mentioned in the introduction.

4 Decision problems on extender sets

4.1 Inclusion of extender sets

Let us consider the following decision problem:

Extender-inclusion	
Input:	An effective subshift $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$, and $u, v \in \mathcal{L}(X)$,
Output:	Whether $E_X(u) \subseteq E_X(v)$

▷ Claim 8. Extender-inclusion is a Π_2^0 problem. Furthermore, restricted on the class of subshifts with computable language, Extender-inclusion is a Π_1^0 problem.

Proof. As: $E_X(u) \subseteq E_X(v) \iff \forall B \in \mathcal{A}^*, u \sqsubseteq B \implies B \notin \mathcal{L}(X) \vee ((B \setminus u) \sqcup v) \in \mathcal{L}(X)$. we obtain the claim: indeed, deciding whether a pattern w belongs in $\mathcal{L}(X)$ is a Π_1^0 problem, and it becomes decidable if X has computable language. ◀

On the other hand, we can prove completeness.

▶ **Proposition 9.** *Extender-inclusion is Π_2^0-complete. On the class of subshifts with a computable language, it is Π_1^0-complete.*

Proof of Π_2^0-hardness for $\mathbb{Z}$ subshifts. We reduce the following known Π_2^0 problem[3]:

Det-Rec-state	
Input:	A deterministic Turing Machine M, and a state q
Output:	Whether q is visited infinitely often by M on the empty input

Let (M, q) an instance of this Det-Rec-state. We construct an effective subshift X on two layers over the alphabet $\{0, 1\} \times \{*, \square\}$ as follows:

- The symbol 1 can only appear above a symbol $*$; and if there are at least two symbols $*$ on the second layer, no symbol 1 can appear on the first layer;
- If there are at least two symbols $*$ on the second layer, at distance, say, $n > 0$, then the configuration is n-periodic; and if M enters q at least n' times, then we impose $n > n'$.

As the rules above forbid an enumerable set of patterns, X is an effective subshift. We then claim that $E_X((0, *)) \subseteq E_X((1, *))$ if and only if M enters q infinitely many times.

Indeed, the symbol $(0, *)$ can be extended either by semi-infinite lines of symbols $(0, \square)$, which are configurations that also extend the symbol $(1, *)$; or by configurations containing n-periodic symbols $(0, *)$, which do *not* extend the symbol $(1, *)$ because of the first rule. However, the second rule implies that for every $n > 0$, this n-periodic configuration exists if and only if M visits q less than n times. This concludes the proof. ◀

Proof of Π_1^0-hardness for computable $\mathbb{Z}$ subshifts. We slightly modify the previous construction to reduce

$$\text{co-Halt} = \{M \mid M \text{ is a Turing Machine that does not stop on the empty input}\}$$

by forbidding periods $< n$ when the machine is still running after n steps. The same argument goes through. ◀

[3] It is a rephrasing of Inf (does a given machine halt on infinitely many inputs?), which is Π_2^0-complete. See [22, Theorem 4.3.2].

4.2 Computing the number of extender sets

Let us determine the computational complexity of the formula $|E_X(n)| \geq k$, given a subshift X and some size n and some k. It is equivalent to the following:

$$\bigvee_{v_1,\ldots,v_k \in \mathcal{L}_n(X)} \bigwedge_{1 \leq i < j \leq k} E_X(v_i) \neq E_X(v_j)$$

As $|E_X(n)| \geq k$ is expressible as finite disjunctions/conjunctions of a Σ_2^0 (even Σ_1^0 if X has computable language) problem over the set $\mathcal{L}_n(X)$, we conclude:

▶ **Lemma 10.** *For an effective subshift X, $|E_X(n)| \geq k$ is a Σ_2^0 formula, and even Σ_1^0 on subshifts with computable language.*

4.3 Upper computational bounds on extender entropies

As $h_E(X) = \inf_n \frac{\log |E_X(n)|}{n^2}$, the following corollary follow from Lemma 10:

▶ **Corollary 11.** *For an effective subshift X, $h_E(X) \in \Pi_3$. For a computable subshift X, $h_E(X) \in \Pi_2$.*

Proof. For fixed X, n, Lemma 10 shows that the set $\{k \leq |E_X(n)|\}$ is a Σ_2^0 set if X is effective (resp. Σ_1^0 set if X has computable language). This means that the $|E_X(n)|$, and therefore $\frac{\log |E_X(n)|}{n^d}$, are Σ_2 (resp. Σ_1) and so $h_E(X) \in \Pi_3$ (resp. $h_E(X) \in \Pi_2$) as the infimum of Σ_2 (resp. Σ_1) real numbers. ◀

5 Realizing extender entropies

5.1 Summary

The fact that sofic and effective subshifts have the same bounds should not come out as a surprise at this point: most of our proofs rely on finding some one-dimensional effective subshift realizing some property, and using Theorem 2 as a black-box: its periodic lift is a $\mathbb{Z}^2$ sofic shift. This section follows the same scheme:

- Section 5.2: we prove in Theorem A that effective $\mathbb{Z}$ subshifts have exactly Π_3 entropies;
- Section 5.3: we lift this construction to $\mathbb{Z}^2$ subshifts and add slight modifications to obtain the same result in Theorem B.

5.2 One-dimensional effective subshifts

Let us focus on one-dimensional subshifts for the time being.

▶ **Theorem A.** *The set of extender entropies of $\mathbb{Z}$ effective subshifts is exactly $\Pi_3 \cap [0, +\infty)$.*

In order to construct a subshift Z_α with $h_E(Z_\alpha) = \alpha$, we would like to have $|E_{Z_\alpha}(n)| \simeq 2^{\alpha n}$. A way to have this would be, as in Section 3, to create subshifts with one extender set per pattern, and with $2^{\alpha n}$ patterns of size n; however, since effective subshifts have Π_1 entropies, this would not realize the whole class of Π_3 numbers.

Yet, realizing the right number of patterns is the main idea behind the proof that follows: we just do not blindly create one extender set per pattern, but only separate extender sets when some conditions are met.

Proof. Let $\alpha \in \Pi_3$ a positive real number, $\alpha = \inf_i \sup_j \inf_k \beta_{i,j,k}$ for some computable sequence $(\beta_{i,j,k})$. We can assume $\alpha \leq 1$, and using [25, Lemma 3.1] we can assume that $(\beta_{i,j,k})$ satisfies some monotonicity properties: for all i, j, $(\beta_{i,j,k})_{k\in\mathbb{N}}$ is (non-strictly) decreasing and converges towards some $\alpha_{i,j}$; for all i, $(\alpha_{i,j})_{j\in\mathbb{N}}$ is (non-strictly) increasing and converges towards some α_i; and the sequence $(\alpha_i)_{i\in\mathbb{N}}$ is (non-strictly) decreasing and converges towards α.

Sketch of the construction We construct a subshift Z_α with several layers:

- The **first layer** encodes an integer i. Intuitively, it tells us which of the α_i we are trying to approximate in this configuration.
- The **second layer** encodes some other integer j. Its purpose is to control the "precision" at which we try to approximate α_i.
- The **density layer** contains a binary word $w_{i,j}$ of density roughly $\alpha_{i,j}$. Note that $\alpha_{i,j}$ is not computable, as it is a Π_1 real number.
- The **free layer** is a way to increase the entropy, by adding "free bits" on top of the 1s of the density layer.

More formally, we first define an auxiliary subshift Z'_α with an extra layer. From Z'_α, we will later define Z_α and prove that $h_E(Z_\alpha) = \alpha$.

Auxiliary subshift Consider this set of five layers:

1. **First layer L_1:** For $A_* = \{*, ␣\}$, we set $L_1 = X_*$ where $X_* = \overline{\bigcup_{i\in\mathbb{N}} \mathrm{Orb}(*␣^{i-1})}$ is the subshift composed of (the closure of) periodic configurations of $*$ symbols separated by ␣ symbols. In other words, denoting $\langle i \rangle_{k_1} \in A_*^{\mathbb{Z}}$ the i-periodic configuration defined by $(\langle i \rangle_{k_1})_p = * \iff p = k_1 \bmod i$, and $\langle \infty \rangle = \{x \in A_*^{\mathbb{Z}} \mid |x|_* \leq 1\}$ the set of configurations having at most a single symbol $*$, we have:

$$X_* = \langle \infty \rangle \cup \bigcup_{i\in\mathbb{N}} \{x \in A_*^{\mathbb{Z}} \mid \exists k_1 \in \mathbb{N}, x = \langle i \rangle_{k_1}\}.$$

2. **Second layer L_2:** We also set $L_2 = X_*$.
3. **Density layer L_d:** Define $A_d = \{0, 1\}$, and define the *density layer* as $L_d = A_d^{\mathbb{Z}}$. Consider the ruler sequence $T = 12131214\ldots$ (see OEIS A001511). Given a real number $\beta \in [0, 1)$ and its proper binary expansion $\beta = \sum_{i\in\mathbb{N}} \beta_i 2^{-i}$, let us define its Toeplitz sequence as:

$$T(\beta) = \beta_{T_0}\beta_{T_1}\ldots\beta_{T_i}\cdots \in \{0, 1\}^{\mathbb{N}}$$

and denote $T(\beta, i)_{k_1} \in A_d^{\mathbb{Z}}$ the i-periodic configuration defined as

$$(T(\beta, i)_{k_1})_p = T(\beta)_{k_1+p \bmod i}.$$

We will use configurations $T(\beta, i)_{k_1}$ to realize specific densities of symbols 1.

▶ Remark. Several densities β can realize the same configuration $T(\beta, i)_{k_1}$. In the rest of the article, when considering a configuration $x = T(\beta, i)_{k_1}$, β will implicitly be the minimal density realizing x.

4. **Free layer L_f:** Let us now add free bits above symbols 1 of A_d. More precisely, denoting $A_f = \{0, b, b'\}$, define the *free layer* as $L_f = A_f^{\mathbb{Z}}$. Given the synchronization function $\pi_{\mathrm{sync}} : A_f \to A_d$ by $\pi_{\mathrm{sync}}(b) = \pi_{\mathrm{sync}}(b') = 1$ (and $\pi_{\mathrm{sync}}(0) = 0$), we say that a pair of configurations $(z^{(d)}, z^{(f)}) \in A_d^{\mathbb{Z}} \times A_f^{\mathbb{Z}}$ are *synchronized* if $\pi_{\mathrm{sync}}(z^{(f)}) = z^{(d)}$.
5. **Periodicity layer L_p:** Denote $A_p = \{\mathsf{p}, \infty\}$ and X_p the finite subshift $L_p = \{\mathsf{p}^{\mathbb{Z}}, \infty^{\mathbb{Z}}\}$.

We now use these five layers to define an auxiliary effective subshift Z'_α:

$$
\begin{aligned}
Z'_\alpha = \Big\{ & (z^{(1)}, z^{(2)}, z^{(d)}, z^{(f)}, z^{(p)}) \in L_1 \times L_2 \times L_d \times L_f \times L_p \mid \\
& (z^{(1)}, z^{(2)} \in \langle\infty\rangle \text{ or } z^{(2)} \in \langle\infty\rangle) \text{ and } \pi_{\text{sync}}(z^{(f)}) = z^{(d)} \Big\} \\
\cup \bigcup_{i,j\in\mathbb{N}} \Big\{ & (z^{(1)}, z^{(2)}, z^{(d)}, z^{(f)}, z^{(p)}) \in L_1 \times L_2 \times L_d \times L_f \times L_p \mid \exists i, j, k_1, k_2 \in \mathbb{N}, \\
& j \geq i,\ z^{(1)} = \langle i \rangle_{k_1},\ z^{(2)} = \langle j \rangle_{k_2},\ \pi_d(z^{(f)}) = z^{(d)},\ z^{(p)} = \mathsf{p}^{\mathbb{Z}}, \\
& \exists \beta \leq \alpha_{i,j},\ z^{(d)} = T(\beta, i)_{k_1} \text{ and } z^{(f)} \text{ is } i\text{-periodic} \Big\}
\end{aligned}
$$

Z'_α is an effective subshift. Indeed, the structure of the configurations is straightforward to enforce, and we need to forbid configurations in which $z^{(1)} = \langle i \rangle_{k_1}$, $z^{(2)} = \langle j \rangle_{k_2}$ and $z^{(d)} = T(\beta, i)_{k_1}$ with $\beta > \alpha_{i,j}$. As $\alpha_{i,j}$ is a Π_1 real number, we can enumerate the densities larger than $\alpha_{i,j}$ and forbid the corresponding patterns.

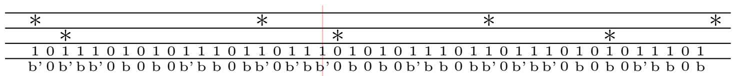


Figure 2 A Type 1 configuration: L_d contains a Toeplitz encoding of $\overline{.1010}^2 = \frac{5}{8}$. $z = (\langle 15 \rangle_{11}, \langle 18 \rangle_1, T(\frac{5}{8}, 15)_{10}, z^{(f)} = (b'0b0b0b'bb0bb'0b'b)^\infty)$. The red line indicates the origin.

The periodicity layer L_p is actually not needed in this proof, but is later used for Theorem B.

The subshift Z_α From Z'_α, we define the effective subshift Z_α by forgetting the L_p component of Z'_α. As the projection of an effective subshift, Z_α is also an effective subshift:

$$Z_\alpha = \pi_{L_1 \times L_2 \times L_d \times L_f}(Z'_\alpha).$$

Denoting the configurations of Z_α by $z = (z^{(1)}, z^{(2)}, z^{(d)}, z^{(f)})$, we separate the configurations of Z_α into two types:

- If the first two layers actually encode integers, *i.e.* if $z = (\langle i \rangle_{k_1}, \langle j \rangle_{k_2}, T(\beta, i)_{k_1}, z^{(f)})$ with $z^{(f)}$ being both i-periodic and synchronized with $T(\beta, i)_{k_1}$ (which forces one extender set per pattern) ; we call z a **Type 1** configuration. In that case, notice that the density β verifies the following inequalities: $\beta \leq \alpha_{i,j} \leq \alpha_i$.
- Otherwise, z is either $z = (\langle\infty\rangle, \langle\infty\rangle, z^{(d)}, z^{(f)})$, $z = (\langle i \rangle_k, \langle\infty\rangle, z^{(d)}, z^{(f)})$ or $z = (\langle\infty\rangle, \langle j \rangle_k, z^{(d)}, z^{(f)})$. In that case, $z^{(d)}$ can be any configuration and $z^{(f)}$'s only constraint is to have b, b' exactly on top of $z^{(d)}$'s symbols 1 (which means that there are very few extender sets). Intuitively, these configurations have no meaning in our construction, and we call them **Type 2** configurations.

We formally prove in Appendix A.1 that $h_E(Z_\alpha) = \lim_i \alpha_i = \alpha$, which concludes the proof. Intuitively, consider two patterns w, w' that can appear in some configurations of Type 1:

- If they differ on their L_1, L_2 or L_d layers, then they have different extender sets.
- Otherwise, they differ on their free bit layer, and they therefore have different extender sets. Indeed, say without loss of generality that $w_0^{(f)} = b$, and $w_0'^{(f)} = b'$. Consider $z \in Z_\alpha$ extending w, with $z^{(1)} = \langle i \rangle_{k_1}$ for some i, k_1. By definition of Z_α, $z^{(f)}$ is i-periodic, and in particular $z^{(f)}|_{i\mathbb{Z}}$ is constant equal to b. This implies that z cannot extend w', as $w_0'^{f} = b'$.

This implies that there is one extender set per pattern appearing in Type 1 configurations. For given n and $i \leq n$, there are roughly $2^{i \cdot \alpha_i}$ patterns of size n in configurations z of Type 1 where $z^{(1)} = \langle i \rangle$. Summing over $i \leq n$ and taking the limit, we obtain $h_E(Z_\alpha) = \lim_n \alpha_n = \alpha$. ◀

5.3 Two-dimensional sofic subshifts

In order to extend Theorem A to multidimensional sofic shifts, an idea could be to generalize the previous construction. Indeed, in the construction of the $\mathbb{Z}$ effective subshift Z_α, **Type 1** configurations that encode an integer i on their first layer have a density (less than or equal to) α_i of free bits which are furthermore i-periodic, making one extender set per such pattern; on $\mathbb{Z}^2$, we could make these free-bits (i,i)-periodic and obtain the same result. Unfortunately, a short argument shows that such a subshift would not be sofic[4].

To solve this issue, we use the same trick that the semi-mirror shift solves in Section 3: instead of "reflecting" the whole half-plane (in our case, the whole $i \times i$ square), reflecting a single bit "non-deterministically" (in our case, a single bit per $i \times i$ square) is enough to separate patterns into distinct extender sets; and having to synchronize a single bit is an amount of information low enough as to make the subshift sofic.

▶ **Theorem B.** *The set of extender entropies of $\mathbb{Z}^2$ sofic subshifts is exactly $\Pi_3 \cap [0, +\infty)$.*

Proof. We use the notations introduced in the proof of Theorem A.

Auxiliary subshift We first define an auxiliary subshift Y'_α as follows. Its layers are:

- **Lifted layers:** We define the first four layers of Y'_α as $L_1^\uparrow \times L_2^\uparrow \times L_d^\uparrow \times L_p^\uparrow$, where L_1, L_2, L_d and L_p are defined in the proof of Theorem A.
- **Free layer L_f:** Still denoting $A_f = \{0, b, b'\}$, we define the free layer as $L_f = A_f^{\mathbb{Z}^2}$. Once again, we define a synchronization function $\pi_{\text{sync}} : A_f \to A_d$ by $\pi_{\text{sync}}(b) = \pi_{\text{sync}}(b') = 1$ (and $\pi_{\text{sync}}(0) = 0$) to synchronize this free layer with the density layer from the lift $Z_{\text{aux}}^\uparrow$: we say that a pair of configurations $(z^{(d)}, z^{(f)}) \in A_d^{\mathbb{Z}^2} \times A_f^{\mathbb{Z}^2}$ are *synchronized* if $\pi_{\text{sync}}(z^{(f)}) = z^{(d)}$.

  ▶ Remark. Note that L_f is not obtained by lifting the L_f layer of the previous proof, as this would lead to a linear instead of a quadratic growth.
- **Marker layer L_m:** We define $L_m = \mathcal{G}$, where $\mathcal{G}$ is the following subshift over the alphabet $A_m = \{\square, \blacksquare\}$: denoting by $[2i]_{m_1,m_2}$ the configuration defined by $([2i]_{m_1,m_2})_{(p_1,p_2)} = \blacksquare \iff (p_1 = m_1 \bmod 2i \ \wedge \ p_2 = m_2 \bmod 2i)$ (in other words, $[2i]_{m_1,m_2}$ draws a square lattice of $\blacksquare$ symbols at distance $2i$ from one another) and $[\infty] = \{x \in A_m^{\mathbb{Z}^2} \mid |x|_{\blacksquare} \leq 1\}$ the set of configurations having at most one symbol $\blacksquare$, we set

$$\mathcal{G} = [\infty] \cup \bigcup_{i \in \mathbb{N}} \{x \in A_m^{\mathbb{Z}^2} \mid \exists m_1, m_2 \in \mathbb{Z}, x = [2i]_{m_1,m_2}\}$$

From these layers, we define Y'_α as follows. Let $Z_{\text{aux}} = \pi_{L_1 \times L_2 \times L_d \times L_p}(Z'_\alpha)$ where Z'_α is the auxiliary subshift defined in the proof of Theorem A. Then:

$$\begin{aligned} Y'_\alpha = \Big\{ (y^{(1)\uparrow}, y^{(2)\uparrow}, y^{(d)\uparrow}, y^{(p)\uparrow}, y^{(f)}, y^{(m)}) \in L_1^\uparrow \times L_2^\uparrow \times L_d^\uparrow \times L_p^\uparrow \times L_f \times L_m \ \Big|\ & \exists i, k_1, m_1, m_2, \\ (y^{(1)}, y^{(2)}, y^{(d)}, y^{(p)}) \in Z_{\text{aux}},\ \pi_{\text{sync}}(y^{(f)}) = y^{(d)\uparrow}, & \\ y^{(1)} \in \langle \infty \rangle \implies y^{(m)} \in [\infty], & \\ y^{(1)} = \langle i \rangle_{k_1} \implies (y^{(m)} = [2i]_{m_1,m_2} \text{ and } & \\ y^{(p)\uparrow} = \mathsf{p}^{\mathbb{Z}^2} \implies y^{(f)}|_{(m_1 + i\mathbb{Z}) \times (m_2 + i\mathbb{Z})} \text{ is constant}) \Big\} & \end{aligned}$$

See Figure 3 for an illustration.

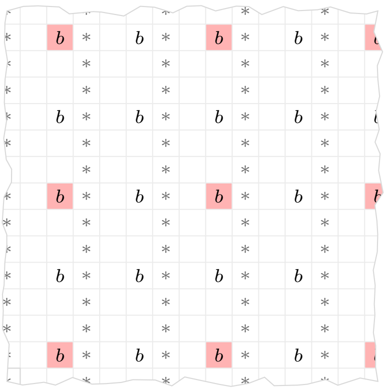


Figure 3 Projection of a (Type 1) configuration on $L_1 \times L_f \times L_m$. The symbols $*$ are on L_1, the symbols b are on L_f and the symbols ■ on L_m. All the other bits of L_f (not drawn here) are free.

In other words, Y'_α is the lift $Z^{\uparrow}_{\text{aux}}$ with two additional layers L_f (containing free bits) and L_m (containing markers ■). We say that $y \in Y'_\alpha$ is of **Type 1** (resp. **Type 2**) if its first four layers $(y^{(1)}, y^{(2)}, y^{(d)}, y^{(p)})$ are the lift of a Type 1 (resp. Type 2) configuration of Z_{aux}.

In a configuration of Type 1 with first and second layers $y^{(1)} = \langle i\rangle^{\uparrow}_{k_1}$ and $y^{(2)} = \langle j\rangle^{\uparrow}_{k_2}$, the density of the layer $y^{(d)}$ is lower than $\alpha_{i,j}$, and in the layer $y^{(f)}$ there is *a single periodic free bit* of period (i,i) whose position modulo (i,i) is determined by the layer $y^{(m)} = [2i]_{m_1,m_2}$. The actual set of configurations of Y'_α is:

$$\begin{aligned}
Y'_\alpha = \Big\{ &(y^{(1)\uparrow}, y^{(2)\uparrow}, y^{(d)\uparrow}, y^{(p)\uparrow}, y^{(f)}, y^{(m)}) \in L_1^{\uparrow} \times L_2^{\uparrow} \times L_d^{\uparrow} \times L_p^{\uparrow} \times L_f \times L_m \;\Big|\\
&y^{(1)}, y^{(2)} \in \langle\infty\rangle \text{ and } y^{(m)} \in [\infty] \text{ and } \pi_{\text{sync}}(y^{(f)}) = y^{(d)\uparrow} \Big\}\\
\cup \bigcup_{i\in\mathbb{N}} \Big\{ &(y^{(1)\uparrow}, y^{(2)\uparrow}, y^{(d)\uparrow}, y^{(p)\uparrow}, y^{(f)}, y^{(m)}) \in L_1^{\uparrow} \times L_2^{\uparrow} \times L_d^{\uparrow} \times L_p^{\uparrow} \times L_f \times L_m \;\Big|\\
&y^{(1)} = \langle i\rangle_{k_1}, y^{(m)} = [2i]_{m_1,m_2}, y^{(2)} \in \langle\infty\rangle, \pi_{\text{sync}}(y^{(f)}) = y^{(d)\uparrow}\\
&y^{(p)\uparrow} = \mathsf{p}^{\mathbb{Z}^2} \implies y^{(f)}|_{(m_1+i\mathbb{Z})\times(m_2+i\mathbb{Z})} \text{ is constant} \Big\}\\
\cup \bigcup_{i,j\in\mathbb{N}} \Big\{ &y^{(1)\uparrow}, y^{(2)\uparrow}, y^{(d)\uparrow}, y^{(p)\uparrow}, y^{(f)}, y^{(m)} \in L_1^{\uparrow} \times L_2^{\uparrow} \times L_d^{\uparrow} \times L_p^{\uparrow} \times L_f \times L_m \;\Big|\\
&j \geq i,\; y^{(1)} = \langle i\rangle_{k_1}, y^{(2)} = \langle j\rangle_{k_2}, y^{(m)} = [2i]_{m_1,m_2}, \pi_{\text{sync}}(y^{(f)}) = y^{(d)\uparrow}, y^{(p)\uparrow} = \mathsf{p}^{\mathbb{Z}^2}\\
&\exists\beta \leq \alpha_{i,j}, y^{(d)} = T(\beta, i)_{k_1} \text{ and } y^{(f)}|_{(m_1+i\mathbb{Z})\times(m_2+i\mathbb{Z})} \text{ is constant} \Big\}
\end{aligned}$$

By Theorem 2, $Z^{\uparrow}_{\text{aux}}$ is sofic. We prove in Appendix A.3 that the subshift Y'_α is a sofic subshift, that is, the extra conditions on the layers L_p, L_f, L_m are sofic.

The subshift Y_α Finally, we define the subshift Y_α by forgetting the L_p component of Y'_α. Contrary to the proof of Theorem A, the layer L_p is actually important here, as it ensures that only Type 1 configurations are forced to have some periodic free bit (more details below). As the projection of a sofic subshift, Y_α is also a sofic subshift:

$$Y_\alpha = \pi_{L_1^{\uparrow}\times L_2^{\uparrow}\times L_d^{\uparrow}\times L_f\times L_m}(Y'_\alpha)$$

[4] Namely, this is the argument that shows that the full-mirror subshift cannot be sofic: there would be $2^{O(i^2)}$ distinct $i \times i$ patterns, but only $2^{(O(i))}$ borders.

We finally claim that $h_E(Y_\alpha) = \alpha$. The proof is very similar to the proof of Theorem A, and the following paragraphs highlight the main modifications:

- Fix w, w' two different $i \times i$ patterns that do not contain a marker ■ and that can both appear in Type 1 configurations of the form $z = (\langle i \rangle_{k_1}^{\uparrow}, \dots)$. If w, w' differ on their first, second or density layers, they have different extender sets. Otherwise, if they only differ on their free bits layer, the forced periodic free bit of Type 1 configurations ensures that they have different extender sets.
  Indeed, assume without loss of generality that w, w' differ in position $(0,0)$. Since w appears in some Type 1 configuration $y = (\langle i \rangle_{k_1}^{\uparrow}, \langle j \rangle_{k_2}^{\uparrow}, T(\beta, i)_{k_1}, y^{(f)}, y^{(m)}) \in Y_\alpha$, and that w does not contain a marker ■, we can pick $y^{(m)} = [2i]_{0+i,0+i}$. Having markers in position $(0 + i \bmod 2i, 0 + i \bmod 2i)$ forces $y^{(f)}|_{(i\mathbb{Z}\times i\mathbb{Z})}$ to be constant[5], *i.e.* to have a periodic free bit; therefore y is not a valid completion of w', as $w'_{(0,0)} \neq y^{(f)}_{(0+i,0+i)}$ since $y^{(f)}_{(0+i,0+i)} = w_{(0,0)}$.
- The layer L_p (that is forgotten after the final projection) is used to prevent Type 2 configurations from forcing (i,i)-periodicity on the free bit marked by L_m. Indeed, there exists Type 2 configurations with marked free bits: $y^{(m)}$ is some $[2i]_{m_1,m_2}$ as soon as $y^{(1)} = \langle i \rangle_{k_1}^{\uparrow}$, even when $y^{(2)}$ actually belongs in $\langle \infty \rangle^{\uparrow}$. In that case, periodicity on the free bits should not be enforced, as it would lead to one extender set per value of the free bits layer, of which there are too many since the density of $y^{(d)}$ can be arbitrary.
  Layer L_p solves this problem: indeed, if $y^{(1)} = \langle i \rangle_{k_1}^{\uparrow}$ and $y^{(2)} \in \langle \infty \rangle^{\uparrow}$, both layers $y^{(p)} = \mathsf{p}^{\mathbb{Z}^2}$ (which forces (i,i)-periodicity of the free bit marked by $[2i]_{m_1,m_2}$) and $y^{(p)} = \infty^{\mathbb{Z}^2}$ (which does not) can exist; after the projection, these two cases are merged and no free bit is forced to be (i,i)-periodic. On the other hands, we force $y^{(p)} = \mathsf{p}^{\mathbb{Z}^2}$ on Type 1 configurations, which (even after projection) enforces (i,i)-periodicity of the free bit marked by $y^{(m)}$.

The complete proof that $h_E(Y_\alpha) = \alpha$ can be found in Appendix A.2. ◀

6 Realizing extender entropies: computable subshifts

The upper-bound from Corollary 11 shows that if X is a $\mathbb{Z}^2$ shift with decidable language, then its extender entropy is a Π_2 real number. We show that the converse holds and obtain:

▶ **Theorem C.** *The set of extender entropies of $\mathbb{Z}^2$ (sofic) subshifts with computable language is exactly $\Pi_2 \cap [0, +\infty)$.*

Sketch of proof. We slightly alter the previous construction.

Let $\alpha = \inf_i \alpha_i = \inf_i \sup_j \beta_{i,j} \in \Pi_2$, for $(\beta_{i,j})$ a computable sequence, while verifying the monotonicity conditions from [25, Lemma 3.1]. We redefine the subshift Z'_α constructed in Theorem A by taking $\beta_{i,j,k} = \beta_{i,j}$ for every $i, j, k \in \mathbb{N}$. The resulting subshift is computable: indeed, the non-computability of $\mathcal{L}(Z'_\alpha)$ came from the fact that, given some $i, j \in \mathbb{N}$ and word $T(\beta, i)$, it was undecidable to know whether $\beta \leq \alpha_{i,j}$. In this new case, since $\alpha_{i,j} = \beta_{i,j}$ is now computable, this becomes decidable. In turn, the subshift now has computable language.

The rest of the proofs, both on $\mathbb{Z}$ and $\mathbb{Z}^2$, goes through in the exact same way. ◀

[5] The layer $y^{(m)}$ voluntarily is $(2i, 2i)$-periodic instead of (i,i)-periodic, since the argument only works whenever w, w' do not contain the ■ marker.

7 Extender sets of minimal subshifts

On minimal subshifts, the problem is much easier and does not even depend on the computability of the language.

▶ **Proposition 12.** *Let X a minimal subshift over $\mathbb{Z}^d$. Then for any $n > 0$ and any patterns $u, v \in \mathcal{L}_n(X)$, $E_X(u) \subseteq E_X(v) \iff u = v$.*

Proof. Let $u, v \in \mathcal{L}_n(X)$ and suppose that $E_X(u) \subseteq E_X(v)$. Then in any configuration, any appearance of u can be replaced by v: if $u \neq v$, we obtain by compactness that forbidding u in X leads to a non-empty proper subshift of X, which contradicts its minimality. ◀

For X a minimal subshift, this implies in particular that $h_E(X) = h(X)$. Since minimal effective subshifts can have arbitrary Π_1 entropy (consider the proof of [16, Theorem 4.77] on computable sequences $(k_n)_{n\in\mathbb{N}}$ of integers), we obtain:

▶ **Corollary 13.** *The possible extender entropies of minimal effective subshifts are exactly the Π_1 real numbers.*

8 Extender sets of subshifts with mixing properties

8.1 Mixing $\mathbb{Z}$ subshifts

Being mixing is a general notion defined on all dynamical systems. We recall here the definition in the case of $\mathbb{Z}$ subshifts: intuitively, being mixing implies that for any pair of admissible words, there exists a configuration containing both of them at arbitrary positions, provided we set them sufficiently far apart.

▶ **Definition 14** (Mixing subshift)**.** *A $\mathbb{Z}$ subshift X is **mixing** if*

$$\forall n > 0, \exists N > 0, \forall u, v \in \mathcal{L}_n(X), \forall k \geq N, \exists w \in \mathcal{L}_k(X), uwv \in \mathcal{L}(X)$$

We say that X is $f(n)$-mixing for some function f if N can be taken equal to $f(n)$ in the previous definition. In the special case where f is constant $f(n) = N$, we simply write that X is N-mixing.

It seemed natural to expect that strong mixing conditions would imply restrictions on the extender entropy of a subshift. Indeed, the examples we mentioned so far either have strong mixing properties (the full shift, $\mathbb{Z}$ SFTs, . . .) and zero extender entropy, or contain strong obstructions to mixing properties (periodicity, reflected positions, . . .) to increase their number of extender sets. Furthermore, similar problems have been studied for topological entropies: *e.g.* [11] shows that some mixing properties (namely, linear block-gluing) do not restrict the possible entropies of multidimensional SFTs, while [10] proves that the set of possible entropies depends on the growth rate of a mixing function of the subshift.

However, we show in this section that even very restrictive mixing properties do not imply anything on the extender entropies.

▶ **Proposition 15.** *Let X a one-dimensional subshift. There exists $X_\#$ a 1-mixing subshift with $h_E(X) = h_E(X_\#)$.*

▶ Remark. If X is sofic (resp. effective), then $X_\#$ can also be taken sofic (resp. effective).

Proof. Let X be a subshift defined by a family of forbidden patterns $\mathcal{F}$ over an alphabet $\mathcal{A}$, and denote $\alpha = h_E(X)$. We can assume that $\mathcal{F} = \mathcal{A}^* \setminus \mathcal{L}(X)$.

Let us define a subshift $X_\#$ over the alphabet $\mathcal{A} \sqcup \{\#\}$ (assuming that $\#$ is a free symbol not in $\mathcal{A}$) by the same family of forbidden patterns $\mathcal{F}$. In other words, configurations of $X_\#$ are composed of (possibly infinite) words of $\mathcal{L}(X)$ separated by the *safe symbol* $\#$. It is clear that $X_\#$ is 1-mixing, as for any $u, v \in \mathcal{L}(Y)$, we have $u\#v \in \mathcal{L}(Y)$. On the other hand, we prove in Appendix B.1 that $h_E(X_\#) = h_E(X)$. ◀

This can be generalized to higher dimensional subshifts, as in the following proposition. There are various mixing notions one can study, and we choose to formulate our results for **block-gluing** subshifts; it is unclear how those could be adapted to other mixing properties.

▶ **Proposition 16.** *For any d-dimensional effective subshift X, there exists a 1-block-gluing effective subshift $X_\#$ with $h_E(X) = h_E(X_\#)$.*

▶ Remark. As opposed to the one-dimensional case, the hypothesis that X is effective is important in this proposition.

For the definition of block-gluingness and the proof of Proposition 16, see Appendix B.2.

9 Reformulation as growth rates of syntactic monoids

In this section, we briefly show a different point on view on the previous results, by relating extender sets with the classical syntactic monoid from the study of formal languages.

For any finite alphabet $\mathcal{A}$ and any language $L \subseteq \mathcal{A}^*$, one can define an equivalence relation, called the **syntactic congruence**, as (see for example [1, Definition 3.6]):

$$\forall u, v \in L, u \sim_L v \iff (\forall x, y \in L, xuy \in L \iff xvy \in L)$$

▶ **Definition 17** (Syntactic monoid). *Let L a language. Then $M(L) = L/\sim_L$ with the concatenation operation is a monoid, called the **syntactic monoid** of L.*

For L a language over $\mathcal{A}$ and $u \in L$, we call **reduced length** of u the non-negative integer $\|u\|_L = \min_{v \sim_L u} |v|$. The **growth rate** of $M(L)$ is then

$$h(M(L)) = \lim_{n\to+\infty} \frac{\log |\{[u] \in M(L) \mid u \in L, \|u\|_L \leq n\}|}{n}$$

For a $\mathbb{Z}$-subshift X, we define its syntactic monoid $M(X)$ as $M(\mathcal{L}(X))$. In this setting, we can reformulate Theorem A as:

▶ **Theorem 18.** *The growth rates of syntactic monoids of effective $\mathbb{Z}$-subshift are exactly the non-negative Π_3 real numbers.*

Sketch of proof. Consider the subshift Z_α from the proof of Theorem A. Then we claim that $M(Z_\alpha)$ has growth rate α. Indeed, for any two words $u, v \in \mathcal{L}_n(Z_\alpha)$ such that $u \not\sim_{\mathcal{L}(Z_\alpha)} v$, we have $E_{Z_\alpha}(u) \neq E_{Z_\alpha}(v)$. So $|\{[u] \in M(X) \mid u \in \mathcal{L}_X(Z_\alpha), \|u\|_{\mathcal{L}(Z_\alpha)} \leq n\}| \leq |E_{Z_\alpha}(n)|$. On the other hand, the argument computing a lower bound on the number of extender sets of Z_α in Appendix A.1 exhibits a family of roughly $2^{n\alpha+o(n)}$ words of $\mathcal{L}_n(Z_\alpha)$ such that any two words of this family cannot be syntactically congruent. This concludes the proof. ◀

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A Proofs of Section 5

A.1 Computations for the proof of Theorem A

Proof of $h_E(Z_\alpha) \leq \alpha$. We simply provide an upper bound of $|E_{Z_\alpha}(n)|$. For a pattern $w \in \mathcal{L}_n(Z_\alpha)$, denoting $w = (w^{(1)}, w^{(2)}, w^{(d)}, w^{(f)})$, we say that w *sees* i (resp. j) at position k if $w^{(1)}$ (resp. $w^{(2)}$) contains two symbols $*$ separated by $i-1$ (resp. $j-1$) symbols $\sqcup$, and $w_k^{(1)} = *$ (resp. $w_k^{(2)} = *$) is the first $*$ in this layer. In that case, any configuration $z \in Z_\alpha$ extending w must have $z^{(1)} = \langle i \rangle_k$ (resp. $z^{(2)} = \langle j \rangle_k$). We now determine the extender sets of the different patterns w.

We partition $\mathcal{L}_n(Z_\alpha)$ in two sets $T_1 \sqcup T_2$. Patterns in T_2 will only appear in configurations of Type 2, and patterns in T_1 can appear in a configuration of Type 1.

As configurations of Type 2 have no restriction on their free layer $z^{(f)}$ or their density layer $z^{(d)}$, the extender set of any $w \in T_2$ is entirely determined by its first two layers (*i.e.* by the pair $(w^{(1)}, w^{(2)})$): since there are poly(n) such pairs, there are at most poly(n) extenders sets of patterns of T_2.

Now, notice that in a configuration z of Type 1, layers $z^{(1)}, z^{(d)}$ and $z^{(f)}$ are i-periodic for some $i \in \mathbb{N}$ and $z^{(2)}$ is j-periodic for some $j \in \mathbb{N}$: this implies in particular that any two distinct patterns $w, w' \in T_1$ have different extender sets. Let us now count the number of such patterns of size n, layer by layer:

1. If $w^{(1)}$ sees i, then it can see values of i ranging from 0 to $n-1$, and different offsets k_1 ranging from 0 to $i-1$; if it does not see i, then there is at most a single symbol $*$ whose position ranges from 0 to $n-1$.
2. If $w^{(2)}$ sees j, then j ranges from i to $n-1$ with offset k_2 ranging from 0 to $j-1$; if it does not see j, then there is at most a single symbol $*$ whose position ranges from 0 to $n-1$.
3. If $w^{(1)}$ sees i, then $w^{(d)}$ is entirely determined by $T(\beta, i)_{[\![0,i-1]\!]}$ for some $0 \leq \beta \leq \alpha_i$. If the first layer does not see i, then $w^{(d)}$ layer is a factor of $T(\beta, i)$ for some $i \geq \lceil n/2 \rceil$ and $0 \leq \beta \leq \alpha_i$. In both cases, for $0 \leq \beta \leq 1$, there are less than $2^{\lceil \log i \rceil}$ different possibilities for $w^{(d)}$.
4. For a fixed $(w^{(1)}, w^{(2)}, w^{(d)})$, we count the number of attributions of free bits in $w^{(f)}$. To proceed, consider whether $w^{(d)}$ can be extended into a configuration $T(\beta, i)$ for some $i \in \mathbb{N}$ and $\beta \leq \alpha_i$ (a pattern could appear in several $T(\beta, i)$ for different values of i; counting the same pattern several times is not a problem when bounding from above):
   a. For each $1 \leq i \leq n$, fixing the attribution of the layer $w^{(f)}$ on a single i-period is enough, since in any Type 1 configuration z extending w with $z^{(d)} = T(\beta, i)$, the layer $z^{(f)}$ will be i-periodic.
      On the other hand, by properties of Toeplitz sequences, there are $i \cdot \beta + O(\log i)$ symbols 1 in an i-period of $T(\beta, i)$. Since the free layer $w^{(f)}$ is synchronized with the density layer $w^{(d)}$, *i.e.* any of these symbol 1 on $w^{(d)}$ can either by b or b' in $w^{(f)}$, there are $2^{i \cdot \beta + o(i)}$ possible attributions for the layer $w^{(f)}$.
      Since $\beta \leq \alpha_i$, we obtain at most $2^{i \cdot \alpha_i + o(i)}$ possibilities for $w^{(f)}$.
   b. If $i > n$, the numbers of 1*s* in a factor of length n in $T(\beta, i)$ is bounded by $n \cdot \beta + O(\log n)$, and $\beta \leq \alpha_i \leq \alpha_n$, so there are at most $2^{n \cdot \alpha_n + o(n)}$ different attributions for the layer $w^{(f)}$.

Then, we obtain for T_1 the following upper-bound:

$$
\begin{aligned}
|T_1| &\leq \sum_{i=1}^{n}\sum_{k_1=0}^{i-1}\sum_{j=i}^{n}\sum_{k_2=0}^{j} 2^{\lceil \log i \rceil}\cdot 2^{i\cdot\alpha_i+o(i)} + \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} 2^{\lceil \log i \rceil}\cdot 2^{n\cdot\alpha_n+o(n)}\\
&\leq \mathrm{poly}(n)\sum_{i=1}^{n} 2^{i\cdot\alpha_i+o(i)}
\end{aligned}
$$

Counting both T_1 and T_2, we obtain:

$$
|E_{Z_\alpha}(n)| \leq \mathrm{poly}(n)\cdot\left(1+\sum_{i=1}^{n}2^{i\cdot\alpha_i+o(i)}\right) \leq \mathrm{poly}(n)\cdot\sum_{i=1}^{n}2^{i\cdot\alpha_i+o(i)}
$$

and a brief computation shows that $h_E(Z_\alpha) = \lim \frac{\log|E_{Z_\alpha}(n)|}{n} \leq \lim \alpha_i = \alpha$. ◀

Proof of $h_E(Z_\alpha) \geq \alpha$. To compute a lower bound, if suffices to observe that for each $n \in \mathbb{N}$, we can find roughly $2^{n\cdot\alpha_n}$ patterns with different extender sets for patterns of length n. Indeed, for $n\in\mathbb{N}$, let $\beta = \alpha_n - 2^{-\lfloor \log n\rfloor}$. Since $\alpha_n = \sup_j \alpha_{n,j}$, there exists some $j_n \geq n$ such that $\alpha_{n,j_n} \geq \alpha_n - 2^{-\lfloor \log n \rfloor} = \beta$.

Now, let $W = T(\beta, j_n)_{[\![0,n-1]\!]}$. Then, the number of symbols 1 in W is bounded from below by $|W|_1 \geq n\cdot\beta - O(\log n) = n\cdot\alpha_n - O(\log n)$.

Now, there are more than $2^{n\cdot\alpha_n - O(\log n)}$ patterns w in $\mathcal{L}_n(Z_\alpha)$ with $w^{(1)} = \langle n\rangle_0 = w^{(2)} = \langle j_n\rangle_0$ and $w^{(d)} = W$ (because of the density layer $w^{(d)}$, that contains either b or b' under each symbol 1 of W). Moreover, each of these patterns w has a different extender set. Indeed, as W is $T(\beta, j_n)_{[\![0,n-1]\!]}$ with $\beta \leq \alpha_{n,j_n}$, each of these patterns can be extended by a configuration of Type 1 $z = (\langle n\rangle_0, \langle j_n\rangle_0, T(\beta, j_n)_0, (w^{(f)})^\infty)$ that is a valid extender for w only because of the i-periodicity of $z^{(f)}$.

In particular, we get that $|E_{Z_\alpha}(n)| \geq 2^{n\cdot\alpha_n - o(n)}$, and we conclude that $h_E(Z_\alpha)\geq\alpha$. ◀

A.2 Computations for the proof of Theorem B

Proof of $h_E(Y_\alpha) \leq \alpha$. Fix $n > 0$. The proof follows the case analysis performed in Theorem A with the same notations.

As in the computation of $h_E(Z_\alpha)$ in the $\mathbb{Z}$ construction, we partition the set $\mathcal{L}_n(Y_\alpha)$ into $T_1 \sqcup T_2$, where T_1 denotes the set of patterns that can appear in a configuration of Type 1 (and T_2 the set of patterns that only appear in configurations of Type 2). Furthermore, we refine the partition of $T_1 \subseteq \mathcal{L}_n(Y_\alpha)$ as follows: denote $T_{1,\blacksquare}$ the patterns of T_1 containing a $\blacksquare$ on their L_m layer, and $T_{1,\square} = T_1 \setminus T_{1,\blacksquare}$ those who do not. We then have $\mathcal{L}_n(X) = T_1 \sqcup T_2 = (T_{1,\square} \sqcup T_{1,\blacksquare}) \sqcup T_2$. We will now bound the number of extender sets for patterns of each family.

- The extender sets of patterns $w \in T_2$ are entirely determined by those of $\pi_{L_1^\uparrow \times L_2^\uparrow}(w) \times \pi_{L_m}(w)$, that is, by its layers L_1 and L_2 on any of its rows, and by the L_m layer. Indeed, for every $w \in T_2$, $(\pi_{L_d}(E_X(w)))^\downarrow$ is a full-shift, so $(\pi_{L_f}(E_X(w)))^\downarrow$ does not depend on $w^{(d)}$; in particular, $E_X(w)$ does not depend on $w^{(d)}$ at all. Similarly, since the layer L_m enforces (i,i)-periodicity of some free bit only in Type 1 configurations (none of those extend w since $w \in T_2$), $(\pi_{L_f}(E_X(w)))^\downarrow$ does not depend on $w^{(f)}$ either, so $E_X(w)$ also does not depend on $w^{(f)}$ at all.
  As L_m is periodic, it is itself completely determined by its period and by the position of the unique $\blacksquare$ in each of its period, so we have $|\pi_{L_m}(T_2)| \leq \sum_{p\leq n} p^2 = \mathrm{poly}(n)$. Since $|\pi_{L_1^\uparrow\times L_2^\uparrow}(T_2)| = \mathrm{poly}(n)$, there are in total $\mathrm{poly}(n)$ extender sets for patterns of T_2.

- The extender sets of patterns of $T_{1,\blacksquare}$ are entirely determined by $\pi_{L_1^\uparrow \times L_2^\uparrow \times L_d^\uparrow}(w) \times \pi_{L_m}(w)$ and by the value of the marked bit in L_f. Indeed, even in Type 1 configurations, all the bits of L_f that are not under a marker $\blacksquare$ are free.
  More precisely, for any $w, w' \in T_{1,\blacksquare}$ with $\pi_{L_1^\uparrow \times L_2^\uparrow \times L_d^\uparrow \times L_m}(w) = \pi_{L_1^\uparrow \times L_2^\uparrow \times L_d^\uparrow \times L_m}(w')$, denote by $(m_1, m_2) \in [\![0, n-1]\!]^2$ a position having a marker, *i.e.* $w^{(m)}_{(m_1,m_2)} = w'^{(m)}_{(m_1,m_2)} = \blacksquare$. Then $E_X(w) = E_X(w')$ if and only if $(w^{(f)})_{m_1,m_2} = (w'^{(f)})_{m_1,m_2}$, that is, if their bit[6] of L_f marked by the $\blacksquare$ are equal. Computations of Theorem A show that $|\pi_{L_1^\uparrow \times L_2^\uparrow \times L_d^\uparrow \times L_m}(T_1)| = \text{poly}(n)$, there are at most poly(n) different extender sets for patterns of $T_{1,\blacksquare}$.
- Finally, we bound the number of extender sets for patterns of $T_{1,\square}$. Observe that a Type 1 configuration extending some pattern of $w \in T_{1,\square}$ must have a layer L_1 equal to $\langle i \rangle_k^\uparrow$ for some i, k, and a periodic $(2i, 2i)$-periodic layer L_m. Therefore, it verifies $i \geq \lceil \frac{n}{2} \rceil$, as otherwise there would be some marker $\blacksquare$ in $w^{(f)}$.
  First, there are at most $\text{poly}(n)$ possibilities for $(w^{(1)}, w^{(2)}, w^{(d)}, w^{(m)})$. Then, $w^{(d)}$ contains a factor of $T(\beta, i)$ for some $i \geq \lceil n/2 \rceil$ and $\beta \leq \alpha_i$; to bound the number of extender sets depending on $w^{(f)}$, we consider the values of i separately (a pattern could appear in several $T(\beta, i)$ for different values of i; counting the same pattern several times is not a problem when bounding from above):
  1. Assume $\lceil n/2 \rceil \leq i \leq n$. The only way to differentiate the extender sets of two patterns of $T_{1,\square}$ by their free layer L_f is for them to be different in some free bit that could be made (i,i)-periodic by putting a marker $\blacksquare$ laying at $(m_1, m_2) \in \mathbb{Z}^2 \setminus [\![0, n-1]\!]^2$ outside of their support. Indeed, all the other positions $(p_1, p_2) \in \mathbb{Z}^2 \setminus ((m1 + i\mathbb{Z}) \times (m2 + i\mathbb{Z}))$ in such a Type 1 configuration y are such that $y^{(f)}_{(p_1,p_2)} = 0$ if $y^{(d)}_{(p_1,p_2)} = 0$, or free to be either b or b' whenever $y^{(d)}_{(p_1,p_2)} = 1$.
     There are i^2 distinct $(m1 + i\mathbb{Z}) \times (m2 + i\mathbb{Z})$ sets of positions for the (i,i)-periodic free bits of a Type 1 configurations. Only $i^2 \cdot \beta + o(i)$ among these i^2 grids contain a symbol 1 in their density layer $w^{(d)}$ ($w^{(d)}$ is $(i,1)$-periodic, so all positions in such grids contain the same symbol), *ergo* only $i^2 \cdot \beta + o(1)$ such grids could actually generate distinct extenders sets.
     Since each of these grids could either contain b or b' in their layer $w^{(f)}$, there are at most $2^{i^2 \cdot \beta + o(i)} \leq 2^{i^2 \cdot \alpha_i + o(i)}$ ways to distinguish these patterns by their layers L_f.
  2. All $i > n$ can be counted at once: if $i > n$, there are at most $\beta \cdot n^2 + o(n)$ positions in $w^{(f)}$ that could be made periodic in a configuration of Type 1 extending w. Since $\beta \leq \alpha_i \leq \alpha_n$ (the sequence $(\alpha_i)_{i \in \mathbb{N}}$ is (non-strictly) decreasing), there are at most $2^{n^2 \cdot \alpha_n + o(n)}$ distinct extender sets.

  So the number of extender sets for patterns of T_1 is at most

$$\text{poly}(n) \cdot \sum_{i=\lceil n/2 \rceil}^{n} 2^{i^2 \cdot \alpha_i}.$$

Combining the extender sets of T_1 and T_2 independently, we conclude with a brief computation that: $h_E(Y_\alpha) \leq \lim_{n \to +\infty} \alpha_n = \alpha$. ◀

Proof of $h_E(Y_\alpha) \geq \alpha$. For the lower bound, it is enough to exhibit some patterns with different extender sets. Basically, we exhibit the two-dimensional equivalents of the patterns

[6] It is enough to look at a single of those marked bits, since in a Type 1 configuration they are all equal.

used in the proof of Theorem A. For $n \in \mathbb{N}$, let $\beta = \alpha_n - 2^{-\lfloor \log n \rfloor}$. There exists some $j_n \geq n$ such that $\alpha_{n,j_n} \geq \alpha_n - 2^{-\lfloor \log n \rfloor} = \beta$.

Let $W = T(\beta, j_n)_{[\![0,n-1]\!]}$. The number of symbols 1 in W is bounded from below by $|W|_1 \geq n \cdot \beta - O(\log n) = n \cdot \alpha_n - O(\log n)$. Now, there are more than $2^{n \cdot \alpha_n - O(\log n)}$ patterns $w \in T_{1,\square} \subseteq \mathcal{L}_n(Y_\alpha)$ with $w^{(1)} \sqsubseteq \langle n \rangle_0^{\uparrow}$ and $w^{(2)} \sqsubseteq \langle j_n \rangle_0^{\uparrow}$ and $w^{(d)} = W^{\uparrow}$. Moreover, each of these patterns w have different extender sets. Indeed, as W is $T(\beta, j_n)_{[\![0,n-1]\!]}$ with $\beta \leq \alpha_{n,j_n}$, each of these patterns can be extended by some configurations of Type 1. Furthermore, for any such two distinct patterns w, w' that only differ on their layer L_f at some position $(m_1, m_2) \in [\![0, n-1]\!]^2$, the configuration $y = (\langle n \rangle_0^{\uparrow}, \langle j_n \rangle_0^{\uparrow}, T(\beta, j_n)_0^{\uparrow}, [2n]_{n+m1,n+m_2}, (w^{(f)})^{\infty})$ extends w but not w'.

Therefore, $|E_{Y_\alpha}(n)| \geq 2^{n^2 \alpha_n - O(\log n)}$, and so $h_E(Y_\alpha) \geq \alpha$. ◀

A.3 Soficity of Y'_α in the proof of Theorem B

Proof of soficity. In order to prove that Y'_α is sofic, we need to show how to construct the two layers L_f and L_m which verify the required conditions.

We will construct a sofic subshift Y_{grid} whose purpose is simply to create a regular grid-like structure, which will be used to synchronize the free bits of the L_f layer. A configuration of Y_{grid} will contain three layers (see Figure 4), which verify for some $i \geq 0$:

- The **column layer** contains a $(i, 1)$ periodic configuration of blue columns.
- The **red grid layer** and the **green grid layer** contain respectively red and green square grids, of mesh $2i$.
- The red and green grids are offset by (i, i); said differently, the red corners lie exactly at the center of the green squares (and reciprocally).

By compactness, configurations containing *e.g.* a single blue column and/or a single corner of a square do exist. In what follows, we denote by A_{grid} the set of symbols used to construct the subshift Y_{grid}.

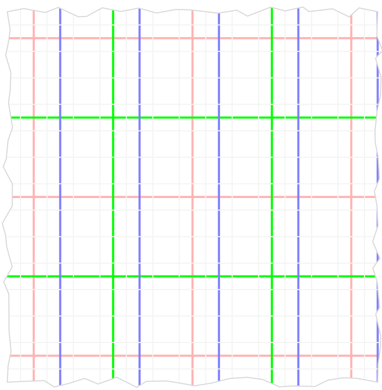

Figure 4 Configuration of Y_{grid}. Vertical blue lines are i-periodic, red and green square grids have mesh $2i$ and are exactly offset by (i, i)

Such a subshift is sofic: each layer taken independently is indeed sofic, placing the two grids correctly relative to one another is , and synchronizing the periods of the three layers can be done using the fact that there needs to be exactly one blue and one green column between two red columns, and one green line between two red lines.

Now consider a subshift Y_{aux} of $Z_{\text{aux}}^{\uparrow} \times Y_{\text{grid}} \times L_f \times L_{\text{sync}}$, where L_f is the full-shift over $\{0, b, b'\}$, and $L_{\text{sync}} = \{0^{\mathbb{Z}^2}, b^{\mathbb{Z}^2}, b'^{\mathbb{Z}^2}\}$, having the following conditions:

- Non-zero symbols of L_f have to be placed exactly at positions containing a symbol 1 of $L_d^\uparrow$ in $Z_{\text{aux}}^\uparrow$.
- Blue columns of Y_{grid} must be placed exactly at positions containing a symbol $*$ in $L_1^\uparrow$ from $Z_{\text{aux}}^\uparrow$. In particular, in a configuration y such that $y^{(1)}$ is some $\langle i\rangle_{k_1}^\uparrow$, this enforces the fact that blue columns in Y_{grid} are also i-periodic, and so the red and green square grids are $(2i, 2i)$-periodic.
- If the layer $L_p^\uparrow$ of $Z_{\text{aux}}^\uparrow$ is constant equal to $\mathsf{p}^{\mathbb{Z}^2}$, then, at all the positions that are corners of the red or green grid of Y_{grid}, we enforce the bit of L_f and of L_{sync} to be the same. As the configuration of L_{sync} is constant, this simply forces some bit of L_f to be (i,i)-periodic. If $L_p^\uparrow$ is $\infty^{\mathbb{Z}^2}$, we enforce nothing.

In particular, in configurations y of Type 2 in Y_{aux} that have $y^{(1)} = \langle i\rangle_{k_1}^\uparrow$, and which therefore have a proper grid on Y_{grid}, the bits of that are marked by the corners of those grids can still be completely free in $y^{(f)}$, as long as $y^{(p)} = \infty^{\mathbb{Z}^2}$.

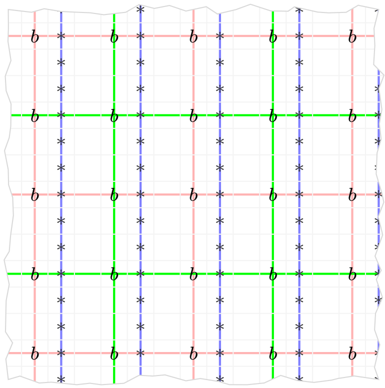

Figure 5 Conditions on Y_{aux} enforced by Y_{grid} and L_{sync}. We obtain Figure 3 by removing the grids and the blue columns, and mapping red corners to ■.

Finally, the subshift Y'_α is the image of Y_{aux} by the following letter-by-letter projection:

$$\phi : A_* \times A_* \times A_d \times A_p \times A_{\text{grid}} \times A_f \times \{0, b, b'\} \mapsto A_* \times A_* \times A_p \times A_d \times A_f \times A_m$$

$$(a_i, a_j, a_d, a_p, a_{\text{grid}}, a_f, b_{\text{sync}}) \mapsto \left(a_i, a_j, a_d, a_p, a_f, \begin{cases} \blacksquare & \text{if } a_{\text{grid}} \text{ is a red corner} \\ \square & \text{otherwise} \end{cases}\right)$$

As a projection of a sofic subshift, Y'_α is also sofic. ◀

B Proofs of Section 8

B.1 $\mathbb{Z}$ mixing subshifts

Proof of Proposition 15. We now prove that $h_E(X_\#) = h_E(X)$. First, we need to introduce the notion of **follower** and **predecessor sets**: for a subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$, its **follower** and **predecessor sets** are respectively defined as $F_X(w) = \{x \in \mathcal{A}^{\mathbb{N}} \mid wx \sqsubseteq X\}$ and $P_X(w) = \{x \in \mathcal{A}^{-\mathbb{N}} \mid xw \sqsubseteq X\}$. In other words, the follower set (resp. predecessor set) of some word w correspond to the set of right-infinite (resp. left-infinite) sequences x such that ux (resp. xu) appears in some configuration of X.

Let $n \geq 0$. We first prove that:

$$|E_X(n)| \leq |E_{X_\#}(n)| \leq |E_X(n)| + \sum_{i+j<n} |P_X(i)||F_X(j)|$$

- The leftmost inequality holds simply because if x extends a pattern $w \in \mathcal{L}(X)$ but not $w' \in \mathcal{L}(X)$, then x also belongs in $X_\#$ and still extends w but not w' in $X_\#$, so that $E_{X_\#}(w) \neq E_{X_\#}(w')$.
- For the rightmost inequality, we need to distinguish some cases according to whether a pattern contains a # or not.
  - Let $w \in \mathcal{L}_n(X_\#)$ that does not contain a symbol #. Then, we claim that $E_{X_\#}(w)$ is entirely determined by $E_X(w)$. Indeed,

    $$E_{X_\#}(w) = E_X(w) \cup \bigcup_{l,r \in \mathcal{A}^* | lwr \in \mathcal{L}(X)} \{(x \# l, r \# x') \mid x, x' \text{ admissible in } X_\#\}$$

    So, for $w, w' \in \mathcal{L}_n(X_\#)$ that do not contain a symbol #, $E_{X_\#}(w) = E_{X_\#}(w')$ if and only if $E_X(w) = E_X(w')$.
  - For patterns $w \in \mathcal{L}_n(X_\#)$ containing at least a symbol #, let $i \leq j$ be the first and last positions in w at which a symbol # appear, and define $l, r \in \mathcal{A}^*$ as $(l, r) = (w_{[\![0,i-1]\!]}, w_{[\![j+1,n-1]\!]})$.
    Then, since # is a safe symbol, we claim that $E_{X_\#}(w)$ is exactly determined by the pair $(P_X(l), F_X(r))$. Indeed,

    $$E_{X_\#}(w) = (P_X(l) \times F_X(r)) \cup \bigcup_{\substack{l',r' \in \mathcal{A}^* | \\ l' \cdot l,\ r \cdot r' \in \mathcal{L}(X)}} \{(y \# l', r' \# y') \mid y, y' \text{ admissible in } X_\#\}$$

  Doing a disjunction on these two cases, and over the pairs $i + j < n$ in the second case (and abusing notations again by denoting $P_X(i) = \{P_X(w) \mid w \in \mathcal{L}_i(X)\}$ and $F_X(j) = \{F_X(w) \mid w \in \mathcal{L}_j(X))$ we obtain:

  $$|E_{X_\#}(n)| \leq |E_X(n)| + \sum_{i+j<n} |P_X(i)||F_X(j)|$$

  As for every n, we have $|P_X(n)| \leq |E_X(n)|$ and $|F_X(n)| \leq |E_X(n)|$, and that $|E_X(n)| = 2^{\alpha n + o(n)}$ (since $h_E(X) = \alpha$), we obtain:

  $$\begin{aligned}|E_{X_\#}(n)| &\leq |E_X(n)| + \sum_{i \leq j} |E_X(i)||E_X(j)| \\ &\leq 2^{\alpha n + o(n)} + \sum_{i \leq j} 2^{\alpha i + o(i)} 2^{\alpha j + o(j)} \\ &\leq \text{poly}(n) \cdot 2^{\alpha n + o(n)}\end{aligned}$$

  Since $|E_{X_\#}(n)| \geq |E_X(n)|$, we conclude that $h_E(X_\#) = \alpha$. ◀

B.2 $\mathbb{Z}^d$ block-gluing subshifts

This section is dedicated to the proof of Proposition 16. First, let us properly define block-gluingness:

▶ **Definition 19.** *Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ any subshift, and $f \colon \mathbb{N} \to \mathbb{N}$ a (non-strictly) increasing function. We say that X is f-**block-gluing** if*

$$\forall p, q \in \mathcal{L}_n(X), \forall k \geq n + f(n), \forall u \in \mathbb{Z}^d, \|u\|_\infty \geq k \implies (p \cup \sigma^u(q) \in \mathcal{L}(X))$$

Said differently, X is f-block-gluing if for any two square patterns of size n, as long as we place them with a gap of size $n + f(n)$ between them, there exist a configuration containing both patterns at any position. As with Definition 14, we will simply write N-block-gluing for constant gluing distance ($f\colon n \to N$).

We prove Proposition 16 in dimension 2 for simplicity, although it works in higher dimension by using hyperrectangles instead of classical rectangles in $\mathbb{Z}^2$.

Sketch of proof of Proposition 16. The proof scheme follows the one of Proposition 15: we define a new subshift $X_\#$ by adding a new symbol #. More precisely, since X is an effective subshift, we can assume that X is defined by the computably enumerable family of forbidden patterns $\mathcal{F} = (\bigcup_{m,n\in\mathbb{N}} \mathcal{A}^{m\times n}) \setminus \mathcal{L}(X)$. Then, we define $X_\#$ as the subshift over the alphabet $\mathcal{A} \sqcup \{\#\}$ as follows: configurations of $X_\#$ consist exactly of (possibly infinite) rectangles containing symbols of $\mathcal{A}$ on a background of symbols #, and the family of forbidden patterns $\mathcal{F}$ is still forbidden (which implies that the interior of any $\mathcal{A}$-rectangle is a globally admissible patterns of X).

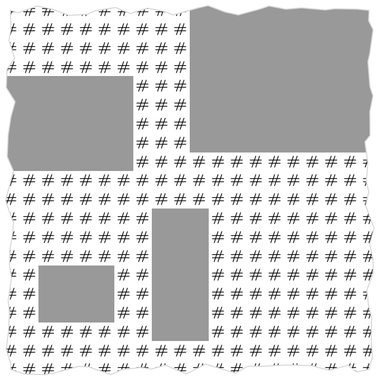

Figure 6 A configuration of $X_\#$. Gray rectangles are admissible patterns of X.

The subshift $X_\#$ is 1-block-gluing: it suffices to "wrap" a $n \times n$ square within a border of # to be able to put any other rectangular pattern next to it. $X_\#$ is also an effective subshift.

We now claim that $|E_X(n)| \leq |E_{X_\#}(n)| \leq 2^{o(n^2)} \cdot |E_X(n)|$.

The left inequality is proved as in the proof of Proposition 15.

For the right inequality, define, for a pattern $w \in \mathcal{L}_n(X)$, its **geometry** $G(w)$: removing the # from w, $G(w)$ is the set of maximal rectangles $r \in \bigcup_{a,b,c,d} [\![a,b]\!] \times [\![c,d]\!]$ that are adjacent to the border of w. To prove the right inequality, we observe that $E_{X_\#}(w)$ is entirely determined by $\{E_X(w|_r) \mid r \in G(w)\}$, and we bound the number of possible $G(w)$ from above:

- First, for a given pattern w, $G(w)$ contains at most $O(n)$ different rectangles adjacent to the border. Indeed, each position in the border of w is either a # or belongs to a single rectangle.
- Each rectangle is specified by two points (if south-west and north-east positions), so that there are $O(n^4)$ such choices, and there are in total $O(n^4)^{O(n)} = 2^{O(n \log n)}$ different possibles geometries $G(w)$ for w ranging in the patterns of support $[\![0, n-1]\!]^2$.
- Finally, for a rectangle $r = [\![a,b]\!] \times [\![c,d]\!]$, denote $\|r\| = (b-a) \cdot (d-c)$ its area and $E_X(r) = \{E_X(w) \mid w \in \mathcal{L}_r(X)\}$. Denoting $\alpha = h_E(X)$, we have $|E_X(n)| = 2^{n^2\alpha + o(n^2)}$

and then:

$$\begin{aligned}
|E_{X_\#}(n)| &\leq \sum_{G(w)|w\in\mathcal{L}_n(X_\#)} \prod_{r\in G(w)} |E_X(r)| \\
&\leq \sum_{G(w)|w\in\mathcal{L}_n(X_\#)} \prod_{r\in G(w)} 2^{\|r\|\alpha+o(\|r\|)} \\
&\leq \sum_{G(w)|w\in\mathcal{L}_n(X_\#)} 2^{\sum\limits_{r\in G(w)} \|r\|\alpha+o(\|r\|)} \\
&\leq \sum_{G(w)|w\in\mathcal{L}_n(X_\#)} 2^{n^2\alpha+o(n^2)} \\
&\leq 2^{n^2\alpha+o(n^2)+O(n\log n)}
\end{aligned}$$

As such, we obtain $h_E(X) \leq h_E(X_\#) \leq h_E(X)$. This concludes the proof. ◀